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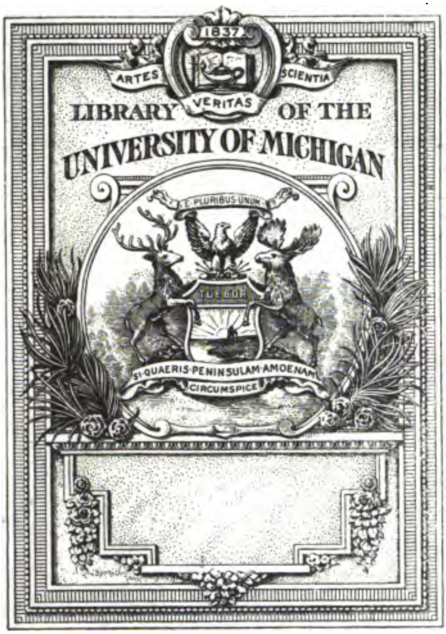
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MECHANICS

AN ELEMENTARY TEXT-BOOK

THEORETICAL AND PRACTICAL

FOR COLLEGES AND SCHOOLS.

DYNAMICS

BY

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FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

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PREFACE.

IT has now come to be generally recognized that the most satisfactory method of teaching the Natural Sciences is by experiments which can be performed by the learners themselves. In consequence many teachers have arranged for their pupils courses of practical instruction designed to illustrate the fundamental principles of the subject they teach. The portions of the following book designated EXPERIMENTS have for the most part been in use for some time as a Practical Course for Medical Students at the Cavendish Laboratory.

The rest of the book contains the explanation of the theory of those experiments, and an account of the deductions from them. This part has grown out of my lectures to the same class. It has been my object in the lectures to avoid elaborate apparatus and to make the whole as simple as possible. Most of the lecture experiments are performed with the apparatus which is afterwards used by the class, and whenever it can be done the theoretical consequences are deduced from the results of these experiments.

In order to deal with classes of considerable size it is necessary to multiply the apparatus to a large extent. The students usually work in pairs and each pair has a separate table. On this table are placed all the apparatus for the experiments which are to be performed. Thus for a class

of 20 there would be 10 tables and 10 specimens of each of the pieces of apparatus. With some of the more elaborate experiments this plan is not possible. For them the class is taken in groups of five or six, the demonstrator in charge performs the necessary operations and makes the observations, the class work out the results for themselves.

It is with the hope of extending some such system as this in Colleges and Schools that I have undertaken the publication of the present book and others of the Series. My own experience has shewn the advantages of such a plan, and I know that that experience is shared by other teachers. The practical work interests the student. The apparatus required is simple; much of it might be made with a little assistance by the pupils themselves. Any good-sized room will serve as the Laboratory. Gas should be laid on to each table, and there should be a convenient water supply accessible; no other special preparation is necessary.

The plan of the book will, I hope, be sufficiently clear; the subject-matter of the various Sections is indicated by the headings in Clarendon type; the Experiments to be performed by the pupils are shewn thus:

EXPERIMENT (1). *To explain the use of a Vernier and to determine the number of centimetres in half a yard.*

These are numbered consecutively. Occasionally an account of additional experiments, to be performed with the same apparatus, is added in small type. Besides this the small-type articles contain some numerical examples worked out, and, in many cases, a notice of the principal sources of error in the experiments, with indications of the method of making the necessary corrections.

These latter portions may often with advantage be omitted on first reading. Articles or Chapters of a more advanced character, which may also at first be omitted, are marked with an asterisk.

I have found it convenient when arranging my own classes to begin with a few simple measurements of length, surface, volume and the like. These are given in Chapter I.

The two following chapters deal with Kinematics and treat the subject in the usual method.

When questions dealing with Momentum, Force, and Energy come to be considered two courses at least are open to the teacher. It is possible to make the whole subject purely deductive; we may start with some definitions and axioms—laws of motion, either as Newton gave them, or in some modern dress—and from these laws may deduce the behaviour of bodies under various circumstances.

Another and more instructive method, it seems to me, is to attempt to follow the track of the founders of Mechanics, to examine the circumstances of the motion of bodies in certain simple cases in the endeavour to discover the laws to which they are subject. This method has been followed in Chapters IV. and V. I have made free use of a piece of apparatus—the ballistic balance—devised by Professor Hicks of Sheffield and by its aid the student is led to realize the importance of momentum in dynamics and to study the transference of this quantity from one body to another. The rate at which momentum is transferred is then considered (Chapter V.) and a name—Force—is given to the rate of

transference. It is shewn that in many cases the rate of change of momentum is constant; while others are referred to in which the rate of change of momentum depends only on the position of the body. Experiments are described to prove that in a given locality all bodies fall with the same uniform acceleration.

It is then shewn that with Atwood's machine, when the rider is on, the weights move with uniform acceleration; and hence the kinematical formulæ obtained earlier in the book relating to the motion of a particle moving with uniform acceleration are verified by experiment; the connexion between the mass moved, the acceleration and the weight of the rider is also investigated.

Some idea of the laws of motion in a simple case having been thus obtained from observation and experiment, Newton's Laws of Motion are enunciated in Chapter VI. and their consequences are deduced in the ordinary way. Some portions of the preceding chapters are of necessity repeated by this method of procedure, which may have other disadvantages as well. These I hope are counter-balanced by the gain resulting from a more intelligent appreciation of the subject on the part of the learner.

Mechanics is too often taught as a branch of pure mathematics. If the student can be led up to see in its fundamental principles a development of the consequences of measurements he has made himself, his interest in his work is at once aroused, he is taught to think about the physical meaning of the various steps he takes and not merely to employ certain rules and formulæ in order to solve a problem.

Chapter VIII. deals with the third law of motion and the principle of energy; while in the succeeding chapters other problems are discussed.

The book has grown considerably beyond the limits of my lectures, though it is by no means a complete treatise on Elementary Mechanics; still I hope it may prove useful as an introduction to the subject.

I have to thank many friends for help. Mr Wilberforce and Mr Fitzpatrick have assisted in arranging and devising many of the experiments. Mr Fitzpatrick has also read all the proofs. Dr Ward's suggestions in many parts of Chapters IV., V., VI. and VII. have been of the highest value. My pupil, Mr G. G. Schott of Trinity College, collected for me many of the Examples, while Mr Green of Sidney College has most kindly worked through all the Examples and furnished me with the answers.

The illustrations have for the most part been drawn by Mr Hayles from the apparatus used in the class.

R. T. GLAZEBROOK.

CAVENDISH LABORATORY.
January, 1895.

The issue of a Second Edition has given an opportunity for the correction of various misprints; in other respects the book is unchanged.

R. T. G.

October, 1895.

CONTENTS.

DYNAMICS.

CHAPTER I.

FUNDAMENTAL QUANTITIES. METHODS OF MEASUREMENT.

1. Mechanics and its signification. Mechanics is defined by Kirchhoff as the Science of Motion. Its object is to describe the kinds of motion which occur in Nature completely and in the simplest manner.

Motion is change of position; that which moves is known as "Matter."

For the complete apprehension of Mechanics the ideas of **Space, Time** and **Mass** are necessary and sufficient.

To these fundamental notions the idea of **Force** depending on the mutual action of bodies is subsidiary.

It will be our first step to consider in some detail the various quantities with which we have to deal and the methods we employ to measure them.

2. Units of Measurement. Any Quantity is of necessity measured in terms of a unit of its own kind; thus we measure the distance between two points in miles or feet, centimetres or inches; the area of a field in square yards or square metres, the mass of a lump of stone in tons or kilogrammes, the time between two events in hours or seconds. We shall thus have to consider, firstly, what are the units in terms of which the various quantities which occur in

Mechanics are to be measured, and, secondly, how we shall compare the quantities with these units.

Thus, for example, when it is stated that the distance between two fixed points is three feet, it is implied that a certain unit of length called a "foot" has been adopted and that three of these placed end to end exactly cover the distance.

3. Fundamental Quantities in Mechanics. The three fundamental quantities in Mechanics in terms of which other quantities which may occur can be measured are, **Length, Time** and **Mass.**

4. The Unit of Length. The unit of length generally used in England is the **Yard.**

Other measures of length, the inch, foot, fathom, mile, etc. are submultiples or multiples of the yard, and can be expressed in terms of it.

The **Yard** is defined by Act of Parliament[1] as follows: "The straight line or distance between the centres of the transverse lines on the two gold plugs in the bronze bar deposited in the office of the Exchequer shall be the genuine standard yard at 62° Fahrenheit, and if lost it shall be replaced by its copies."

In accordance with the Weights and Measures Act of 1878 the British Standards are now kept at the Standards' Office of the Board of Trade at Westminster. The copies referred to above are those preserved at the Royal Mint, the Royal Society, the Royal Observatory, and the Houses of Parliament.

Another unit of length which is authorized by Act of Parliament in England is the **Metre.**

This was defined by a law of the French Republic in 1795 to be the distance between the ends of a rod of platinum made by Borda, the temperature of the rod being that of melting ice. At this date the distance between the pole and the equator along a certain meridian arc of the Earth's surface

[1] 18 and 19 Vict. cap. 72, July 30, 1855.

had recently been measured by Delambre, and it was supposed that Borda's platinum rod represented one ten-millionth of this distance.

Further research has shewn that this is not exactly the case, and thus the metric standard of length is not the terrestrial globe but Borda's platinum rod.

The divisions of the metre are decimal. Thus

10 Decimetres = 1 Metre.

10 Centimetres = 1 Decimetre.

10 Millimetres = 1 Centimetre.

It is this fact and not the actual length which gives the metric system its value for scientific measurements. In such measurements the unit of length is now almost invariably the **Centimetre,** that is to say, it is one-hundredth part of the length of Borda's platinum rod when at the temperature of melting ice.

The relation between these two standards, the yard and the metre, has been the subject of very careful investigation. According to the most recent measurements it has been found that

$$1 \text{ metre} = 1{\cdot}09362 \text{ yards}$$
$$= 39{\cdot}37079 \text{ inches.}$$

Hence

$$1 \text{ inch} = 0{\cdot}0253995 \text{ metre}$$
$$= 2{\cdot}53995 \text{ centimetres.}$$

In this book we shall adopt the **Centimetre** as the unit of length.

5. Methods of measuring Lengths. The measurement of a length consists in the determination of the number of centimetres and fractions of a centimetre which are contained in it, and the method to be adopted in making the measurement will depend to some extent on the magnitude of the length; different methods would be required to measure a fraction of a centimetre or many kilometres. Some of the methods used in measuring small lengths will be given later.

6. Measurement of Area and of Volume. The units of area and of volume depend directly on that of length; they are respectively a square whose side is one centimetre, and a cube whose edge is one centimetre: in measuring an area we determine the number of square centimetres it contains, in measuring a volume we find the number of cubic centimetres in it. The volume of 1000 cubic centimetres is called a Litre, and is often employed as a unit of volume.

7. Experiments on Measurement of Length, Area and Volume.

EXPERIMENT 1. *To explain the use of a vernier and determine the number of centimetres in half a yard.*

You are given a rod half a yard long and a metre scale divided to centimetres. The scale has a vernier attached. This is another short loose scale which has ten divisions marked on it. On laying this along the metre scale it will be found (as in Fig. 1) that these ten divisions occupy the

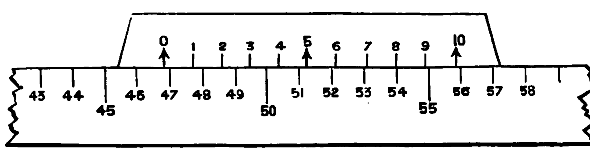

Fig. 1.

same length as nine divisions of the scale. Each division is therefore $\frac{9}{10}$ of a division of the scale, so that one division of the vernier is less than one of the scale by $\frac{1}{10}$ of a scale division, that is, in this case, by 1 millimetre. Place the rod so that one end coincides exactly with the end division of the scale, and place the vernier along the scale so that its end division, marked by an arrow, coincides with the other end of the rod. This division will probably not come opposite to a division of the scale. Suppose it falls between two, say 46 and 47. The rod is between 46 and 47 centimetres long. The vernier enables us to measure the exact length more

nearly; for on looking along the vernier it will be seen that its divisions and those of the scale get more and more nearly coincident until some division of the vernier, say the eighth, coincides almost exactly with one of the scale. Let us count back from this to the arrow-head or division 0 of the vernier, remembering that a vernier division is 1 mm. less than a scale division. The distance between 7 of the vernier and the corresponding scale division is 1 mm., between 6 and the scale division 2 mm., and so on, so that the distance between 0 and the scale division, which we have supposed to be 46, is 8 mm. The rod is therefore 46·8 cm. long. Had the coincidence of vernier and scale been at 5 or 6 the rod would have been 46·5 or 46·6 cm. We have merely to note the division of the vernier which coincides with a scale division and remember that in this case, when 10 vernier divisions coincide with 9 scale divisions, the divisions of the vernier enable us to read to tenths of the scale divisions. Other examples of the vernier should be studied, such as one in which twenty divisions correspond to nineteen of the scale, which therefore reads to twentieths of a scale division.

EXPERIMENT 2. *To find the circumference of a circular disc and so to verify the formula* Circumference $= 2\pi \times$ Radius *where π stands for $3\frac{1}{7}$ approximately.*

Measure the diameter of the disc by laying it on a finely divided scale, or better by the use of the calipers (Fig. 2)—a

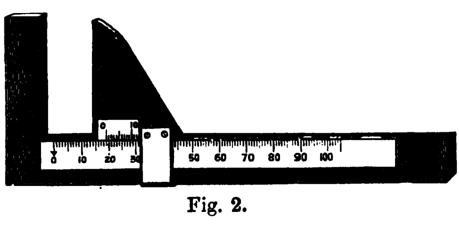

Fig. 2.

pair of calipers for the purpose can be constructed out of a draughtsman's T-square and set square. A scale is marked along the straight edge of the T-square and a vernier on the set square. Lay the T-square on the table and place the

disc so as to touch both the straight edge and the cross piece of the square. Place the set square against the straight edge and slide it along until it touches the disc. Since now both the cross piece of the T-square and one edge of the set square are at right angles to the straight edge of the former the distance between these two as measured along the scale and vernier will give the diameter of the disc. To find the circumference; make a mark with a pencil, or otherwise, on the edge of the disc, and place this in contact with the zero of a finely divided scale, then roll the disc along the scale, taking care that it does not slide, until the mark again comes in contact with a division of the scale. Note this division. The distance between this division and the zero of the scale is equal to the circumference of the disc. Repeat the observations to secure accuracy. It will be found that the ratio of the circumference to the radius is approximately equal to $2 \times \frac{22}{7}$. This ratio is usually denoted by 2π. Thus $\pi = \frac{22}{7}$ approximately.

EXPERIMENT 3. *To find the area of a circle and to verify the formula* Area $= \pi r^2$, *where* r *denotes the radius of the circle.*

You are given a sheet of paper divided by two series of parallel lines at right angles to each other into a number of small squares[1]. The distance between any two consecutive lines is $\frac{1}{10}$ inch, so that each square has an area of ·01 sq. inch. Draw on this a circle of some 3 or 4 inches diameter and measure the diameter. The circle will enclose a large number of complete squares. Count these up and reckon their area. The circumference will also intersect a number of squares. Estimate for such intersected squares the total area which lies within the circle. Thus the area of the circle can be found approximately and the formula verified. This method can be applied to any other figure.

EXPERIMENT 4. *To find the volume of a sphere and to verify the formula* Volume $= \frac{4}{3}\pi r^3$, *where* r *is the radius of the sphere.*

[1] Such paper can be obtained from Messrs Waterlow & Co., London Wall.

Measure the diameter of the sphere by the calipers and hence find its radius. Place the sphere in a test tube or small beaker, Fig. 3, which has a mark made on its outside by means of a file or by gumming on a piece of paper. Fill a burette with water up to a known volume, and let the water run from the burette into the beaker until the latter is filled up to the mark, and note at what level the water in the burette now stands. Find hence the volume of water which has been placed in the beaker. Remove the sphere and the water from the beaker, and by again letting water run in from the burette find the volume of the beaker up to the mark. The difference between these two volumes is clearly the volume of the sphere, and the formula can be verified. The volume of any other solid which sinks in water can be found in the same way. Care must be taken to remove all air bubbles from the solid.

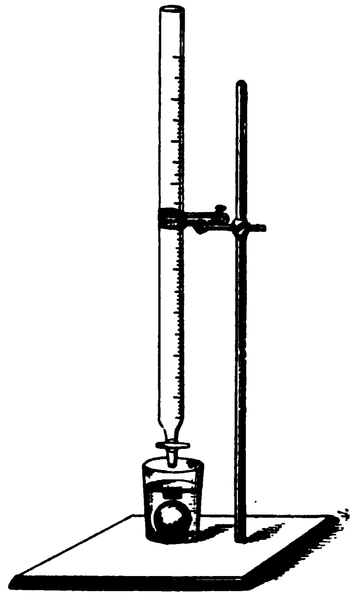

Fig. 3.

EXPERIMENT 5. *To find the thickness of a glass cover-slip by the screw.*

The instrument, Fig. 4, consists of a platform with three feet, whose extremities form an equilateral triangle. Through the centre of the platform passes a fourth foot, which can be raised or lowered by means of a screw.

The pitch of the screw used is 20 threads to the inch, so that if the platform in which the screw works be held firm and the screw turned once round, its end advances or recedes $\frac{1}{20}$ of

an inch. A disc is fixed on to the screw-head and its edge divided into 100 parts, and a vertical scale divided into $\frac{1}{20}$ inch is attached to the platform. If the disc be turned so as to bring the edge of this scale from any one division to the next, the end of the screw moves one-hundredth of one-twentieth of an inch, or $\frac{1}{2000}$ inch; hence, by noting the number of whole turns and parts of a turn made by the disc, we can measure the distance moved over by the screw-point. The whole number of turns are given by the readings of the vertical scale, for the disc moves over one division for each turn. Place the instrument on a flat sheet of glass, and turn the disc until the screw-point is in contact with the glass, read the screw-head; place five or six cover-slips one on the top of the other on the glass and raise the screw until they will just pass underneath it. Read the position of the disc when the point of the screw is just in contact with the top cover-slip, having noted the number of whole turns made by it. From this whole number of turns and the two readings of the disc calculate the total thickness of all the cover-slips, and then, by dividing this by the number of slips, the average thickness of each one.

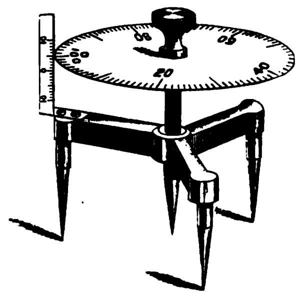

Fig. 4.

EXPERIMENT 6. *To use the screw gauge to measure the diameter of a wire.*

The Screw Gauge is shewn in Fig. 5. It consists of a metal arm ABC; through one end of this passes a steel plug D with a planed face, and through the other a screw EF. The pitch of the screw is half a millimetre, and the end E is planed so as to be parallel to the

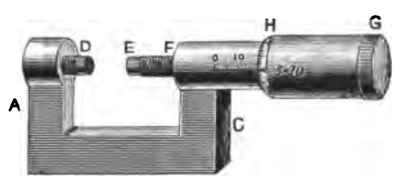

Fig. 5.

face D and perpendicular to the axis of the screw. The screw can be turned by means of the milled head G until its end E comes in contact with D; each complete turn of the screw separates the planed faces by half a millimetre. A scale of half-millimetres is engraved on the frame of the instrument parallel to the axis of the screw and the milled head G carries a cap H with a bevelled edge. The circumference of this edge is divided into a scale of fifty parts, and when the end of the screw is in contact with D the zero of this scale and the zero of the scale on the frame should coincide. On making a complete turn of the screw the cap is moved back half a millimetre, and the zero mark on the bevelled edge is brought opposite the first division of the linear scale. Thus the divisions of this scale which are exposed register the number of half-millimetres between D and E. Since the bevelled edge is divided into fifty parts a rotation through a single part corresponds to a separation of the plane ends by $\frac{1}{50}$ of $\frac{1}{2}$ of a millimetre or by $\frac{1}{100}$ of a millimetre. Thus, if a division (say 24) of the bevelled edge coincides with the linear scale, the distance between the plane faces is a whole number of half-millimetres, which is given by the number of divisions of the linear scale exposed, together with $\frac{24}{100}$ of a millimetre. Thus, to measure the distance between the plane ends, read the number of half-millimetres exposed on the linear scale and add to this the number of hundredths of a millimetre given by the reading of the scale on the bevelled edge.

When using the instrument to measure the diameter of a wire first test the zero reading; then hold the wire between D and E and turn the screw-head G until the wire is gently clipped between the two plane faces. In this case the distance between these faces is the diameter of the wire.

8. Other Instruments for measuring lengths.

(*a*) *Scales and Compasses.*

A pair of compasses and a finely divided scale are often the most convenient apparatus for measuring lengths. The compasses are adjusted until the distance between their points is exactly the length to be measured; they are then applied to the scale and the length is read off. Instead of an ordinary scale a diagonal scale may be used. This is shewn in Fig. 6. There are eleven equidistant parallel lines running the whole length of the scale dividing it into ten spaces. The scale is

divided into inches by lines running across it at right angles to this series of parallel lines. These are numbered **0, 1, 2, 3,** starting from D as zero. The first inch, AB and CD, along each of the top and bottom lines

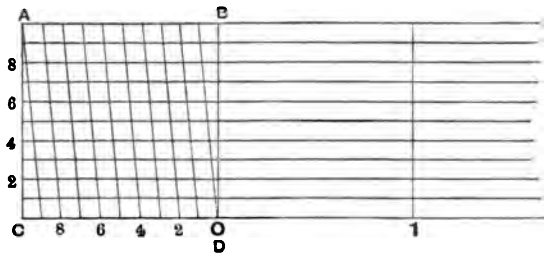

Fig. 6.

is divided into tenths. The alternate divisions, starting from D to the left, are numbered 2, 4, 6 etc., the alternate vertical divisions starting from C upwards are also numbered. Lines are drawn obliquely across the first rectangular division $ABDC$ of the scale as shewn in the figure; thus A is joined to division 9 of CD, division 9 of BA to division 8 of CD and so on; these oblique lines enable us to measure to one-tenth of the small divisions of the scale. For consider the distance between any vertical line, say that through the **1** inch division, and an oblique line such as that through the small division 4. When measured along the lowest horizontal line this distance is 1·4 inches, when measured along the top line it is 1·5 inches. Thus on passing from the bottom to the top of the scale it increases by ·1 inch, but it increases by an equal amount for each vertical space passed over, and here are ten of these spaces, hence the increase for each vertical space is ·01 inch. Thus the distance along the fifth line from the bottom and between the vertical line through the 1 inch mark and the oblique line through the ·4 inch mark is 1 inch + 4 tenths + 5 hundredths or 1·45 inches.

Thus to measure on the scale a distance with the compasses place the right leg on one of the vertical divisions at the point where it crosses the bottom horizontal line, say at division **2** inch. Let the left leg of the compass fall between the points in which the fourth and fifth oblique lines cut the bottom horizontal line. Then the distance is between 2·4 and 2·5 inches. Slide the compasses upwards, keeping the right-hand leg in the vertical division through **2**, and the line joining the two legs parallel to the horizontal lines on the scale, until the left-hand leg falls on a horizontal line as close as possible to the point in which it is intersected by one of the oblique lines; let this occur on the fifth horizontal line; the distance between the legs is greater than 2·4 inches by 5 hundredths of an inch; thus it is 2·45 inches.

(*b*) Caliper-Compasses are made in special forms for measuring the dimensions of curved bodies. Thus, Fig. 7 shews a pair of Calipers of simpler construction than the slide calipers described in EXPERIMENT 2.

The *outside calipers AB* can be set so that the points *A*, *B* just come in contact with two points on the outside of a cylindrical or convex surface, the distance between which is required, while by means of the *inside calipers CD* the distance between points on the inside of a cavity within which the instrument can be introduced can be measured; in either case the distance is found by adjusting the calipers and then laying off the length between the points on a scale.

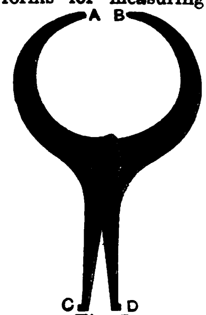

Fig. 7.

(*c*) *The Beam Compass.* This instrument is shewn in Figure 8. A sliding piece *C*, fitted with a vernier and a clamping screw, is attached

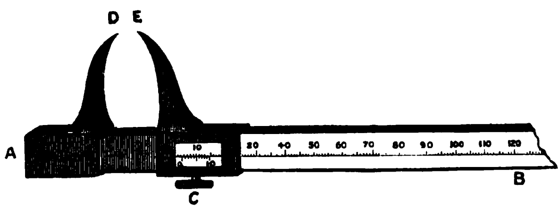

Fig. 8.

to a long straight scale *AB*. A point *D* is attached to one end *A* of the scale and the sliding piece *C* carries a similar point *E*.

The instrument is adjusted so that when the points *D* and *E* are in contact the vernier is at zero on the scale; the reading then of the scale and vernier in any position gives the distance between the points *D* and *E*. The instrument is set so that *D* and *E* coincide respectively with the two points the distance between which is required, and this distance can then be read off directly[1].

9. Time. The next fundamental Physical Quantity which we have to consider is **Time**. "The idea of Time," says Maxwell, "in its most primitive form is probably the recognition of an order of sequence in our states of consciousness." We can

[1] For further particulars as to the method of using such measuring instruments, see Glazebrook and Shaw, *Practical Physics*, Chapter IV.

associate certain sense expressions in a group and separate them off from other groups which we perceive *simultaneously*. Thus we gain the idea of space, but we have also the power of perceiving things *in succession*, we recognize a group of sense impressions as like another distinct group, the impress of which is stored in our memory; we perceive events which follow each other as well as others which have a simultaneous existence.

Our measure of time is derived from the apparent motion of the stars; this apparent motion is a consequence of the motion of the earth round its axis, and we feel that this motion, in the interval from noon to noon, marks off series of like sequences of events; we recognize that the time occupied in one such complete rotation is approximately constant. Owing to the motion of the earth round the sun the interval between two successive passages of the sun across the meridian of any place differs slightly from day to day. The average of such intervals during the year is the mean solar day.

A mean solar day contains 86400 seconds and the fundamental unit of time is the **Mean Solar Second.**

10. Mass. Our third fundamental quantity is called Mass.

If we consider the bodies with which we have to deal as composed of Matter, then any body will consist of a definite quantity of matter. This quantity is usually called its **Mass.**

We shall find however in the sequel that we can give a definite meaning to the term Mass as used in Mechanics without attempting to define the term Matter. We have means for comparing with great accuracy the masses of different bodies; we can therefore measure the Mass of any body in terms of some standard Mass. For the present then we look upon **Mass** as a property of bodies which we recognize by experiment and which we can only define when we have considered those experiments.

We do not know what matter is, it may be that it has no phenomenal existence apart from our conception of it, but it is beyond our province to discuss this here. If we assume that there is a substratum of something we call matter in a body, then the quantity of that matter is measured by the mass of the body, and the masses of bodies can be compared in an exact manner.

There is one case in which there is no difficulty in comparing the quantity of matter in two bodies. For consider two cubes of some homogeneous[1] substance such as platinum, each one centimetre in edge, they are alike in all respects; if they are composed of matter the quantities they contain are obviously equal. The two cubes combined will have double the volume of either singly; there is double the quantity of matter in the two that there was in either. *The quantities of matter in two portions of the same homogeneous substance are proportional to the volumes of the two.* We cannot apply this argument to portions of different substances; equal volumes of iron and lead we shall see have different masses, they are said to contain different "quantities of matter"; it is only when we have considered the laws of motion that we can state exactly what is meant when we assert that the "quantities of matter" in two given bodies are equal, and how it is possible to compare the "quantity of matter" in a lump of iron with that in a heap of feathers.

11. Measurement of Mass. Masses are measured in terms of a unit of mass.

DEFINITIONS. *The mass of a certain lump of Platinum marked PS* 1844 1 *lb., deposited in the Standards department of the Board of Trade at Westminster, is the English Unit of Mass and is called the* **Pound Avoirdupois.**

The mass of a certain lump of platinum made by Borda in 1795 *and kept at Paris is the Unit of Mass on the metrical system and is called the* **Kilogramme.**

It was intended that Borda's kilogramme should be equal to the mass of 1 litre or 1000 cubic centimetres of distilled water at 4° C.[2] The exact determination of the mass of such a volume of water is difficult and its value probably differs

[1] A *homogeneous* substance is one which has identical properties at all points. Water or any other liquid, glass, brass, iron are examples of homogeneous substances. Substances, such as a piece of conglomerate rock, which have different properties at different points, are called *heterogeneous*.

[2] The mass of water which can be contained in 1000 c.c. is greater at this temperature than at any other, hence this temperature was chosen as the standard. See Glazebrook, *Cambridge Natural Science Manuals, Heat*, p. 88.

slightly from that of Borda's platinum mass. Hence the metrical standard of mass is really the mass of a lump of platinum and not, as was intended, the mass of a definite number of units of volume of water.

Still the statement that the mass of 1 cubic decimetre (1000 c.c.) of distilled water is 1 kilogramme is sufficiently nearly true for most purposes, and enables us to introduce great simplification into many numerical calculations.

The unit of mass which is now usually adopted for scientific purposes is the **Gramme. One Gramme** *contains one-thousandth part of the mass of Borda's kilogramme Standard.*

Since a kilogramme (1000 grammes) is very approximately the mass of 1000 c.c. of distilled water, a gramme is very approximately the mass of 1 c.c. of distilled water at 4° C.

The divisions of the gramme are decimal:

10 Decigrammes = 1 Gramme,
10 Centigrammes = 1 Decigramme,
10 Milligrammes = 1 Centigramme.

Careful experiment has shewn that the kilogramme contains 2·2046 pounds.

12. Density. For a given substance, the mass of a body depends on its volume, while, for bodies of given volume, the mass depends on the substance of which the bodies consist and on its physical state. A large lump of iron has a greater mass than a small lump of iron, but a small lump of iron may be of greater mass than a large lump of cork. It is useful to have some term to denote the mass of a definite volume of any body, say 1 cubic centimetre.

DEFINITION OF DENSITY. *The* **Density** *of any homogeneous substance is the mass of unit volume of that substance.*

It follows from this definition that to determine the density of a body we must find the number of units of mass in the unit of volume, we require therefore to know the unit of mass and the unit of volume, if these be the gramme and the cubic centimetre respectively, we may say that the density is so

many grammes per cubic centimetre. Thus in these units the density of water is 1 gramme per c.c., that of iron 7·76 grammes per c.c. In any other units the numerical measures of the densities of these substances would differ from the above. Thus a cubic foot of water contains 998·8 oz. or 62·321 lb.; hence the density of water is 998·8 oz. per cubic foot or 62·321 lb. per cubic foot; iron is 7·76 times as dense as water, hence its density is $7·76 \times 62·321$ lb. per cubic foot.

From the above definition of density we can find a relation between the mass, the volume and the density of a body.

PROPOSITION 1. *To shew that if the mass of a homogeneous body be* M *grammes, its density* ρ *grammes per cubic centimetre and its volume* V *cubic centimetres, then* $M = V\rho$.

For by the definition,

$$\begin{aligned} \text{the mass of } 1 \text{ c.c.} &= \rho \text{ grammes,} \\ \text{therefore the mass of } 2 \text{ c.c.} &= 2\rho \text{ grammes,} \\ \text{and the mass of } 3 \text{ c.c.} &= 3\rho \text{ grammes,} \\ \text{hence the mass of } V \text{ c.c.} &= V\rho \text{ grammes.} \end{aligned}$$

Therefore $$M = V\rho.$$

We may write this as

$$\rho = \frac{M}{V},$$

and thus we have the result that the density of a homogeneous substance is the ratio of its mass to its volume.

A result similar to the above holds for any other consistent system of units.

Various methods of determining by experiment the density of a body will be given later[1].

13. The Comparison of Masses. A balance is the instrument usually employed in the comparison of masses. The theory of the balance is discussed in the Statics, and the method of measuring mass is there considered.

[1] See Hydrostatics.

14. The C.G.S. system of measurement. We shall find that the other physical quantities with which we have to deal in Mechanics can be expressed in terms of the units of length, time and mass or of some of these units. When we take the Centimetre, the Gramme and the Second as fundamental units we are said to employ the C.G.S. system. This is now generally used for scientific purposes; when a quantity has been measured in terms of these fundamental units, it is said to have been determined in absolute measure.

If we know the relation between the C.G.S. system and some other system of units it is easy to change from one to the other in our calculations. Thus if we wish to change to the Foot-Pound-Second System we have approximately

$$1 \text{ cm.} = \cdot 03281 \text{ feet.}$$
$$1 \text{ gramme} = \cdot 002205 \text{ pound.}$$

Examples. (1). *Find the number of cubic feet in a litre.*

$$\begin{aligned} 1 \text{ litre} &= 1000 \text{ c.cm.} \\ &= 1000 \times (\cdot 03281)^3 \text{ c. feet.} \\ &= \cdot 03532 \text{ c. feet.} \end{aligned}$$

(2). *The density of a piece of glass is 2·5 grammes per c.cm., find it in lb. per c. foot.*

We have

$$\begin{aligned} 1 \text{ c.cm.} &= (\cdot 03281)^3 \text{ c. feet} \\ &= \cdot 00003532 \text{ c. feet} \\ 2\cdot 5 \text{ grammes} &= 2\cdot 5 \times \cdot 002205 \text{ lb.} \end{aligned}$$

Hence a volume of ·00003532 c. feet contains $2\cdot 5 \times \cdot 002205$ lb.

Thus density required

$$= \frac{4\cdot 5 \times \cdot 002205}{\cdot 00003532} \text{ lb. per c. foot.}$$

And this reduces to $2\cdot 5 \times 62\cdot 43$ or 156·08 lb. per c. foot.

15. Terms used in Mechanics. Mechanics is the Science of Motion. In studying motion we shall generally require to know both the **Displacement** or change in position of the body, and also the time during which that displacement has occurred.

This branch of the subject is called **Kinematics.** It may be described as the Geometry of Motion; Geometry deals with Space only, Kinematics has for its subject Space and Time.

When we come to consider the mutual relations between moving bodies the science of motion is called **Kinetics**; while in **Dynamics** we pay special attention to the connexion between Force and Motion. In **Statics** we consider the conditions which must exist among a set of Forces impressed on a body which remains at rest.

Statics and Dynamics are usually applied to the Mechanics of Solid bodies. The Sciences which deal with the equilibrium and motion of Fluid bodies are respectively **Hydrostatics** and **Hydrodynamics.**

We are concerned in nature with Material Bodies. A **Body** is a portion of "matter" bounded in every direction. We shall consider a Body as composed of a number of material Particles.

A **Material Particle** is a portion of "matter" so small that for the purposes of our investigations the distances between its different parts may be neglected.

In Dynamics we deal first with the motion of one or more isolated particles, or of a body which we can treat as a particle; we can afterwards proceed to consider the motion of a body of finite size.

We must remember that it will depend on circumstances whether we can treat a body as a particle or not. Thus, apart from a small effect due to the resistance of the air, the shape of a falling stone does not affect the rate at which it falls to the earth; we may solve a problem relating to a falling stone correctly on the supposition that the whole of the stone is concentrated into one point and that the stone behaves as a particle; the same would be true of a cricket ball so long as it is in the air, but the motion of the cricket ball, on striking the ground or being hit by the player, depends on its shape and on the amount of spin given to it by the bowler; these must be considered before we can state how it will move immediately after it is struck, in solving this part of the problem we cannot treat the ball as merely a particle. The words "for the purposes of our investigations" in the above definition are of importance; in the present book when considering Dynamics we deal with the motion of particles.

EXAMPLES.

MENSURATION.

[Take the value of π as 22/7.]

1. Reduce to centimetres (1) 1 ft. 2 in., (2) 2 yds., (3) 5 ft., (4) 1 furlong.

2. Reduce to kilometres (1) 1 mile, (2) 4000 miles, (3) 600 yds., (4) 1 metre, (5) 25 millimetres.

3. Find in centimetres the circumference of circles whose radii are (1) 1 ft., (2) 10 yds., (3) 4000 miles, (4) 750 metres.

4. Find in square centimetres the areas of the circles whose radii are given in Question 3.

5. A circle of radius 5 inches is cut out from a circular disc of radius 9 inches; find the area of the remainder.

6. The circumference of a circle is 1 mile; find its area.

7. The area of a circle is equal to that of a rectangle whose sides are 44 and 126 feet; find its radius.

8. A circle and a square have the same perimeter; determine which has the greater area.

9. An equilateral triangle is described on one side of a square of which the side is 10 feet; find the area of the figure thus formed.

10. A circle of 20 centimetres radius is divided into three parts of equal area by two concentric circles; find the radii of the circles.

11. A circle is 20 centimetres in radius; find the area of a square which can be inscribed in it.

12. A sphere when placed in a beaker as in Exp. 4 displaces 38·786 cubic centimetres of water; find its radius and its surface.

13. Ten cover-slips are placed under the spherometer as described in Experiment 5, the pitch of the screw being ½ a millimetre, and the point is raised 8·56 turns; find the average thickness of a cover-slip.

14. The density of copper is 8·95 grammes per c. cm. The diameter of a piece of copper wire is 1·25 mm. and its length 1025 cm.; find its mass.

15. Find the density of a cylinder 1 foot in height and 6 inches in radius whose mass is 60 lbs.

16. Find the density of a sphere 10 cm. in radius and 5 kilogrammes in mass.

17. Determine the density of the cylinder described in Question 15 in grammes per c. cm.

18. Find the density of a pyramid on a triangular base each side of which is 10 cm. and which has an altitude of 30 cm., the mass of the pyramid being 8 kilogrammes.

19. The density of mercury is 13·59 grammes per c. cm.; find it in grains per cubic inch.

20. Compare the densities of a sphere 5 cm. in radius, 5 kilos. in mass, and a cylinder 1 foot in height, 6 inches in radius and 60 lb. in mass.

CHAPTER II.

KINEMATICS. VELOCITY.

16. Motion. A Body is said to move when it is in different positions at different times. Thus, in order to determine the motion of a body, we have to determine its position at different times and investigate whether the position changes or not. We may notice first that the motion with which alone we can deal is relative motion.

Two passengers seated in a railway carriage are at rest relatively to each other and to the carriage; they are however in motion relatively to objects by the side of the line along which the train is moving. The planets are in motion relatively to the sun; the whole solar system, sun, planets and satellites, is in motion relatively to the stars. In all problems of motion we must have some point which so far as that motion is concerned we treat as fixed and from which we regard the motion. We may investigate the motion of a cricket ball thrown upwards from the earth, but in the investigation we should usually suppose the point from which the ball is thrown to be at rest; as a fact of course this is not true, that point and the ball with it partake of the motion of the earth round its centre, of the motion of the earth's centre round the sun, and so on, but for our purposes this is immaterial.

DEFINITION. Motion *is change of position.*

17. Measurement of Position of a Particle. The position of one particle relatively to another is determined

if we know the *length* and the *direction* of the line joining the two. We may say then that one particle, A Fig. 9, is in motion relatively to a second particle B whenever the length or direction of the line AB joining the two varies.

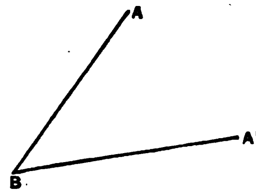

Fig. 9.

When a particle is moved from one position A to another position A' it is said to be displaced. The Displacement of the particle is measured by the length and direction of the line AA'.

We are however in general concerned with the rapidity with which the change in position occurs; moreover the motion may be *Uniform* or *Variable*.

DEFINITION. *The Motion of a particle is* **Uniform** *if the particle passes over equal spaces in equal times.*

The motion of a particle is **Variable** *if the particle passes over unequal spaces in equal times.*

Since an interval of time is measured by the angle which the earth turns through about its axis in that interval, equal times are those in which the earth turns through equal angles. If then in a series of intervals in which the earth turns through equal angles a moving particle passes over equal distances the motion of the particle is uniform; if on the other hand the distances traversed by the particle are unequal, while the angles turned through by the earth are equal, then the motion is variable.

18. Rate of Change of a Quantity. The phrase, **Rate of**, is one which will often occur and which it is desirable to consider. By *rate of change* is meant generally the change in a quantity which takes place during some given interval of time adopted for convenience as the unit of time—or more exactly the ratio which that change bears to the interval of time during which it has occurred.

Thus the statement that the Rate of Interest is 3 per cent. per annum means that in *one year* a sum of £100 increases by £3. If this rate continues uniform, then in 10 years the increase on £100 will be 3×10 or £30; the total increase is obtained by multiplying the rate of interest by the time during

which interest has accrued. Again, if we know that in 5 years £100 has increased by £15 and that the rate of interest has been uniform, we infer that that rate is 15/5 or 3 per cent. per year.

We might speak in the same way of the rate of growth of the population of a town, meaning the increase in population per week or per day as the case may be, or of the daily rate of progress of a building, meaning the amount built in a day.

Thus we see (1) Any quantity varies uniformly when it increases—or decreases—by equal amounts in equal times, and (2) the rate of change of a quantity, which varies uniformly, is the ratio of the change in that quantity to the interval of time during which it has occurred; it is measured by the change which takes place in the unit of time.

19. Average Rate of Change. But Quantities do not always change *uniformly*. The amount of interest obtainable for a given sum may vary from day to day. The daily rate of growth of the population of a town will not be the same throughout the year; more children are born on some days than on others. In such cases we are often concerned with the **Average**[1] **Rate of Change.** This average rate of change is found by calculating the actual change during any time and dividing that by the time.

Thus the statement that during the year the average daily rate of increase in the population of London was 340, does not mean that 340 children were born on each day, but that during the year 340×365 or 124100 were born, so that the total increase during the year is the same as though 340 were born on each day. Or again, consider the case of a railway train which performs a journey of 42 miles in an hour. We should say that its average rate of motion is 42 miles an hour, but this does not imply that it is moving uniformly at the rate which would enable it to traverse the distance in this time; at times it moves more quickly, at the stations it is at rest for some few minutes, and on starting again its rate of motion is less than 42 miles per hour. We must distinguish

[1] The word average as used here has reference to time.

between its *average* rate of motion and its rate of motion *at* any instant of the hour.

20. Graphical Representation of Rate of change. Suppose now we represent on a diagram the two cases of uniform and variable motion thus. Draw a horizontal line, say 30 cm. long, to represent an hour; divide it into half centimetres so that each 5 mm. represents 1 minute, and at the end of each division erect a vertical line to represent the distance traversed up to the end of that minute. Then, in the case of uniform motion, if we represent 1 mile by 1 cm., since 42 miles per hour is the same as 42/60 or ·7 miles per 1 minute, the first vertical line will be ·7 cm. or 7 mm. long, the second will be twice this, the third three times, and so on; the vertical lines will increase uniformly in height, each will be 7 mm. higher than the preceding; a line joining these ends will be straight and will be uniformly inclined to the horizon. The figure obtained will be similar to that shewn in Fig. 10.

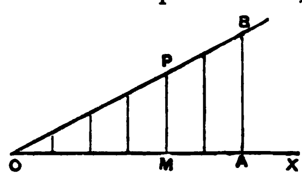

Fig. 10.

21. Variable Rate of Change.

Consider now the case in which the rate of motion is not uniform; let us suppose for the present that we may treat it as uniform during each minute, but that it varies from one minute to the next; the increments in the lengths of the various vertical lines will be different. The train starts slowly, the first line will therefore be less than 7 mm. long, the second less than 14 mm. After a time however the train must exceed its average rate of motion, each successive increment will be greater than 7 mm. until another station is approached, when they will again decrease; during the three or four minutes for which the train stops the corresponding vertical lines will all be of the same length, the

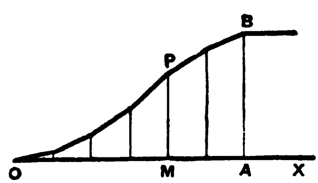

Fig. 11.

line joining their ends will be horizontal. The diagram obtained will resemble Fig. 11. The rate of change of the train's position during different minutes is variable; unequal distances are traversed in equal times; the rate of change however *for each minute* is obtained by finding the alteration during that minute; moreover, if we multiply the change so obtained by 60, the number of minutes in the hour, we can get the rate of motion in miles per hour.

Now the figure just obtained does not represent accurately the motion of the train, for it does not move uniformly for 1 minute, then change its rate of motion and so on; we should get a nearer representation to the truth if we divided each 5 mm. into 60 parts, each representing a second, and supposed the rate of motion to be uniform for each second but to change at the end of every second; this would not be exact, but by proceeding thus and dividing each second into a very large number of very small fractions and supposing the rate of motion uniform during each fraction, but variable from fraction to fraction, we may get as close a representation to the truth as we please. In this manner we come to see that *the Rate of Motion at any moment is found by dividing the distance traversed during an indefinitely short interval of time by the number of seconds in that interval.*

If we divide each minute as shewn in Figs. 10 or 11 into a very large number of parts we shall obtain, instead of the series of straight lines, such as those shewn in Fig. 11, a very much larger number of such straight lines. Each of these will be very short, and it will be impossible to distinguish the many-sided polygon thus formed and a figure bounded by a regular smooth curve as in Fig. 12. The one however represents the series of discontinuous changes at very brief intervals, the other the regular continuous change in the rate of motion which actually occurs with a train. We may compare the two processes with those of mounting a hill (a) by a series of stairs

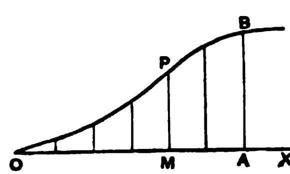

Fig. 12.

or steps, (*b*) by a gradual slope. If the number of steps in the stair be sufficiently large and the size of each sufficiently small the two processes are indistinguishable, the steps merge into the continuous slope. We arrive then at the definition of the term Rate of Change applicable to all cases.

DEFINITION OF RATE OF CHANGE. *The* **Rate of Change** *of any quantity is the ratio of the change in that quantity to the interval of time during which it has occurred when that interval is sufficiently small*[1].

The rate of change therefore is found by dividing the change which has taken place during an interval of time by that interval when the interval is made sufficiently small.

Let us now suppose that at a given moment the rate of change ceases to vary and that the quantity concerned continues to change at the same rate as it did at that moment. From this time on the rate of change is uniform, and is equal therefore to the ratio of the change in *any* interval—not necessarily a very small interval—to the interval during which that change has taken place. If we take the interval to be one second—the unit of time—we may in this case say that the rate of change is the change which has taken place in one second; or the change per second.

We may thus sum up with the statement that the *Rate of change of any quantity, when uniform, is the change in that quantity per second, and, when variable, is the ratio of the change which would take place in that quantity to the interval of time during which that change has occurred when that interval is made very small.* In measuring the rate of change of any quantity we adopt the process indicated in the definition above and divide the change occurring in any interval by the number of seconds in the interval when this number is sufficiently small.

22. Velocity and its measurement. We proceed now to apply the idea of rate of change to various dynamical quantities.

[1] This definition it will be seen is the same as the statement on page 21, except that the last clause is additional. This clause is not necessary if the rate of change be uniform.

DEFINITION. **Velocity** *is Rate of Change of Position.*

We have already defined Motion as change of position, we may therefore state that Velocity is rate of motion.

Velocity may be either *Uniform* or *Variable*. **Uniform Velocity** is measured by the change in position which occurs per second. **Variable Velocity** is measured by the ratio of the change in position in a given interval to the number of seconds in that interval when that number is sufficiently small, that is by the change which would occur per second if during the second the velocity remained uniform.

A particle moving with *uniform* velocity describes equal spaces in equal times. A particle moving with *variable* velocity describes unequal spaces in equal times.

To measure velocity we need to know the change in position, or displacement, per second; this change is determined if we know (1) the distance the particle moves through, (2) the direction of motion. The word **Speed** is employed to denote the distance traversed per second without reference to the direction.

DEFINITION. *The* **Speed** *of a particle is the rate at which it describes its path.*

A particle moves with uniform **Speed** if it travels over equal distances in equal intervals of time; if the **Velocity** be uniform, not merely will the distances be equal but their directions will be the same.

In strictness therefore the velocity of a particle can be uniform only when the particle is moving in a straight line and passes over equal distances in equal times. The term uniform velocity is however often applied where uniform speed would be more accurate; thus the hand of a clock is said to move with uniform velocity, but since the direction of motion changes the velocity is not strictly uniform though the speed is.

23. Motion of a particle with uniform speed.

PROPOSITION 2. *To find the distance—* s *cm.—traversed in* t *seconds by a particle moving with a uniform speed of* v *cm. per second.*

Since the speed is uniform, the particle passes over v cm. in each second.

Hence distance traversed in 1 second $= v$ cm.,
distance traversed in 2 seconds $= 2v$ cm.,
distance traversed in 3 seconds $= 3v$ cm.,
....
distance traversed in t seconds $= tv$ cm.

Therefore $$s = vt.$$

Hence also $$v = \frac{s}{t}.$$

This last result gives a formal proof of the statement that speed when uniform is measured *either* by the distance traversed per second *or* by dividing the distance traversed in any interval of time by that interval.

24. Average speed. The **Average speed** of a particle moving over a given distance in a given time may be defined as the speed with which a second particle, moving uniformly, would describe the given distance in the given time.

It is found therefore by dividing the distance traversed by the number of units of time taken to traverse it.

Thus, consider two trains, one of which is moving uniformly at the rate of 50 miles an hour, while the second starts from one station and arrives, after an interval of an hour, at a second 50 miles distant. The *average* speed of the second train is *the* speed of the first train.

25. Variable speed. The actual speed of the second train at each moment is not 50 miles an hour. To find its value we should require to determine the distance traversed by the train in some very short interval, and divide that distance by the interval; probably in the case of a train the speed would not vary much during a single second, and if we could measure accurately the distance traversed in one second we should determine the speed during that second.

26. Units of speed. Speed is measured by the number of units of length traversed per unit of time.

The numerical measure therefore of the speed or of the velocity of a particle will depend on the unit of length

and on the unit of time. The same speed may be expressed by very different numbers.

Thus, since a mile contains 5280 feet, a speed of 60 miles per hour is the same as one of 60×5280 feet per hour; and since an hour contains 3600 seconds this velocity is the same as one of $60 \times 5280/3600$, or 88, feet per second. In order then to specify a speed or a velocity we must state the unit of length and the unit of time.

A velocity of 88 means one which is 88 times as great as the unit of velocity, and conveys no definite information unless we state clearly what that unit is. It may be a velocity of 88 feet per second, or 88 miles per hour, or anything else.

DEFINITION. *A particle has* **Unit Velocity** *when it traverses unit distance in unit time.*

In the C.G.S. system, a particle has unit velocity when it traverses 1 centimetre per second.

In stating, then, that the velocity of a particle is v, it is necessary to specify the unit distance and the unit time, and to write, if the C.G.S. system be used, a velocity of v cm. per sec.

Examples. (1). *Which train has the greater speed, one moving at the rate of* 60 *miles an hour or one which travels* 100 *yards in three seconds?*

The first train in 60×60 seconds moves over

$$60 \times 1760 \text{ yards};$$

$$\therefore \text{ in 1 second it moves } \frac{1760 \times 60}{60 \times 60} \text{ yards};$$

$$\therefore \text{ the speed is } 29\tfrac{1}{3} \text{ yards per second.}$$

The second train in 1 second moves $\frac{100}{3}$ yards;

$$\therefore \text{ the speed is } 33\tfrac{1}{3} \text{ yards per second.}$$

Thus the second train is moving 4 yards per second the faster.

(2). *Find in feet per second the speed of the earth round the sun, assuming it to describe in 365 days a circle of 92000000 miles radius.*

Taking the value of π as 22/7 the circumference of the earth's orbit is

$$2 \times 22 \times 92000000/7 \text{ miles,}$$

or

$$2 \times 22 \times 1760 \times 3 \times 92000000/7 \text{ feet.}$$

Again, 365 days contain $365 \times 24 \times 3600$ seconds.

Hence in $365 \times 24 \times 3600$ seconds the earth moves

$$\frac{2 \times 22 \times 1760 \times 3 \times 92000000}{7} \text{ feet;}$$

$$\therefore \text{ the speed is } \frac{2 \times 22 \times 1760 \times 3 \times 92000000}{7 \times 365 \times 24 \times 3600} \text{ feet per second.}$$

And this reduces to 97691 feet per second.

(3). *Find the average velocity of a train which takes 7 minutes to traverse the first three miles after leaving a station, then moves for half an hour at the rate of 40 miles an hour, and finally comes to rest, taking 5 minutes to traverse the last 2 miles.*

The whole distance travelled is $3+20+2$ or 25 miles. The time taken is $7+30+5$ or 42 minutes. Thus the average speed is 25/42 or about ·595 mile per minute.

27. Graphical representation of Velocity.

Proposition 3. *To shew that a velocity can be represented by a straight line.*

To determine the value of a velocity we require to know its amount and its direction; these two quantities can be represented by a straight line containing as many units of length as the velocity contains units of velocity and drawn in the direction of motion.

Thus velocities may be completely represented by straight lines.

28. The Composition of Displacements. Consider a man in a railway carriage. Let him move diagonally across the carriage from A to B, Fig. 13. Then if the carriage is at rest his displacement is AB, but suppose the carriage to be in motion and let AA' represent the displacement of any point of the

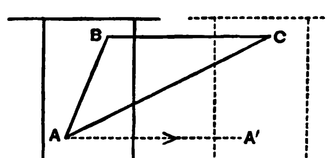

Fig. 13.

carriage during the interval in which the man has moved from A to B. Draw BC equal and parallel to AA'; then relatively to the carriage the man moves from A to B, but relatively to the rails B has moved from B to C, thus the man is at C; his actual displacement[1] is AC, and it is made up of the displacement AB relative to the carriage and BC relatively to the lines.

In this case AC is said to be the *Resultant* of the two displacements AB and AA', and these displacements are spoken of as the *Components* of AC.

Moreover we may continue this process. The rails are not at rest, they are in motion round the axis of the earth.

Let AA'', Fig. 14, represent the displacement of any point on the rails. Draw CD equal and parallel to AA''; then both man, carriage and rails have been displaced through a distance represented by CD; the man therefore will be at D. His displacement is AD, and this is the resultant of AB, BC and CD while these displacements are the components of AD.

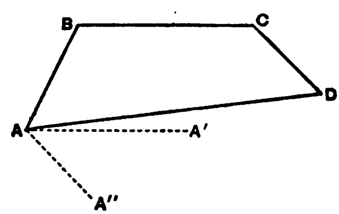

Fig. 14.

DEFINITION. *The single displacement which is equivalent to two or more displacements impressed on a particle is called the* **Resultant** *of those displacements.*

Each of a number of individual displacements, the combined effect of which is equivalent to a single displacement, is spoken of as a **Component** of that single displacement.

PROPOSITION 4. *To find the resultant of a number of displacements.*

[1] This does not at all imply that the man has moved along the straight line AC, but merely that he was at A and is at C.

Consider in the first place two displacements OA and OA', Fig. 15. Draw AB equal and parallel to OA'. In consequence of the displacement OA alone the particle would be at A, in consequence of the second displacement OA' the point A is brought to B. Thus the particle is brought to B and its resultant displacement is OB.

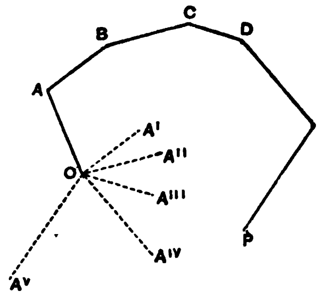

Fig. 15.

Now let there be three displacements OA, OA', OA'', construct the figure as above and draw BC equal and parallel to OA''. In consequence of the displacement OA'', B is brought to C, thus the particle is at C and its displacement is OC.

The general rule, therefore, is obvious. From any point O draw OA to represent the first displacement from A, the extremity of OA, draw AB to represent the second, from B draw BC to represent the third, and so on. Thus if P be the last point thus found OP is the Resultant displacement.

We notice that the various displacements and their resultant form a closed polygon; if it should happen that the point P should coincide with O it is clear that the resultant displacement is zero; the particle will remain at rest. Thus if a series of displacements can be represented by the sides of a closed polygon taken in order the particle remains at rest. Moreover it is immaterial in what order the displacements are made; we can prove this graphically by drawing the figure in various ways, starting with OA' or OA'' instead of OA, then drawing from A', $A'B'$ to represent OA, and so on; it will be found that we always arrive in the end at the same point P.

In general then, if the several displacements be represented by all but one of the sides of a polygon taken in order their resultant is represented by that side taken in the opposite

direction. This proposition is called the polygon of displacements.

29. Special cases of the composition of displacements.

PROPOSITION 5. *When the component displacements are all in the same straight line the resultant is their algebraical sum.*

For consider two such displacements. Draw OA, Fig. 16, to represent the first, AB to represent the second; then AB is in the same straight line as OA and if OA and AB are drawn in the same direction, Fig. 16, then $OB = OA + AB$.

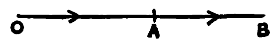

Fig. 16.

While if OA and AB are drawn in opposite directions, Fig. 17, then

$$OB = OA - AB.$$

Fig. 17.

In either case OB is the *algebraical* sum of OA and AB.

The proposition may clearly be extended to three or more displacements.

30. The Parallelogram of Displacements.

PROPOSITION 6. *If two displacements represented in direction and magnitude by two straight lines* OA, OB *meeting at a point be impressed on a particle, the resultant is* OC, *the diagonal through* O *of the parallelogram which has* OA, OB *for adjacent sides.*

For from A draw AC, Fig. 18, equal and parallel to OB. Join OC and BC. Then $OACB$ is a parallelogram and OC is the diagonal through O.

In consequence of the displacement OA the particle is moved to A; in consequence of the displacement OB, A is moved to C. Thus OC is the resultant

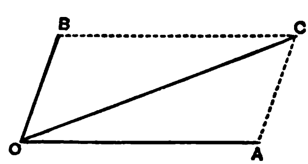

Fig. 18.

displacement and it is the diagonal through O of the parallelogram $AOBC$.

This proposition may be put into a slightly different form, thus:

If two sides OA, AC of a triangle OAC, Fig. 19, represent displacements impressed on a particle, then the third side OC represents the resultant displacement. In this form it is known as the Triangle of Displacements.

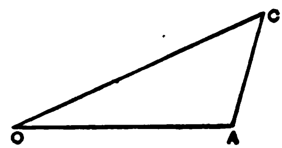

Fig. 19.

Proposition 7. *To find an expression for the resultant of two displacements at right angles.*

Let OA, OB, Fig. 20, represent two displacements, P, Q respectively, at right angles to each other. Complete the rectangle $AOBC$. Let R be the resultant of P and Q, then R is represented by OC.

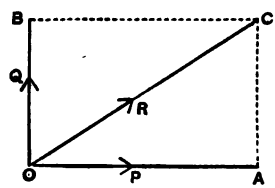

Fig. 20.

Since the angle OAC is a right angle we have

$$OC^2 = OA^2 + AC^2$$

$$= OA^2 + OB^2,$$

$$\therefore\ R^2 = P^2 + Q^2.$$

Hence $$R = \sqrt{\{P^2 + Q^2\}}.$$

*Proposition 8. *To find an expression for the resultant of two displacements inclined to each other at any angle.*

Let OA, OB represent respectively two displacements P, Q inclined to each other at an angle γ.

Complete the parallelogram $AOBC$. Then OC represents R the resultant of P and Q. Draw CD perpendicular to OA meeting OA produced, Fig. 20 (a), or OA, Fig. 20 (b), in D. Then $AOB = \gamma$; in Fig. 20 (a) the angle γ is less than a right angle; in Fig. 20 (b) it is greater.

Now in Fig. 20 (a),

$$OD = OA + AD = OA + AC\cos DAC$$
$$= OA + OB\cos AOB = P + Q\cos\gamma,$$
$$CD = AC\sin CAD = OB\sin\gamma = Q\sin\gamma.$$

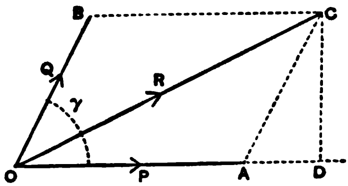

Fig. 20 (a).

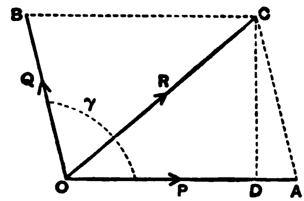

Fig. 20 (b).

In Fig. 20 (b),

$$OD = OA - AD = OA - AC\cos DAC$$
$$= OA - OB\cos(180 - \gamma) = OA + OB\cos\gamma$$
$$= P + Q\cos\gamma,$$
$$CD = AC\sin CAD = OB\sin(180 - \gamma) = Q\sin\gamma.$$

Hence in either case we have

$$R^2 = OC^2 = OD^2 + DC^2$$
$$= (P + Q\cos\gamma)^2 + Q^2\sin^2\gamma$$
$$= P^2 + Q^2 + 2PQ\cos\gamma;$$
$$\therefore\ R = \sqrt{\{P^2 + Q^2 + 2PQ\cos\gamma\}}.$$

There are many special cases of this last proposition which can be solved by Geometry without reference to Trigonometry. Thus, suppose the angle between the two displacements to be 45°.

Hence, constructing Fig. 21 as above, we have

$$AD^2 + CD^2 = AC^2 = Q^2.$$

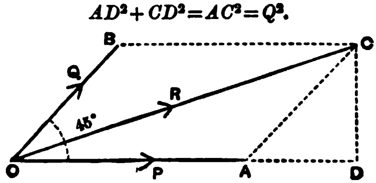

Fig. 21.

Also
$$AD = DC;$$
$$\therefore\ AD = DC = \frac{Q}{\sqrt{2}}.$$

And
$$OD = OA + AD = P + \frac{Q}{\sqrt{2}}.$$

Hence
$$R^2 = OC^2 = OD^2 + DC^2$$
$$= \left(P + \frac{Q}{\sqrt{2}}\right)^2 + \frac{Q^2}{2}$$
$$= P^2 + Q^2 + PQ\sqrt{2}.$$

Or again, if $\gamma = 60°$, we have, Fig. 22,
$$AD = \tfrac{1}{2}AC = \tfrac{1}{2}Q,\quad CD = \frac{Q\sqrt{3}}{2}.$$

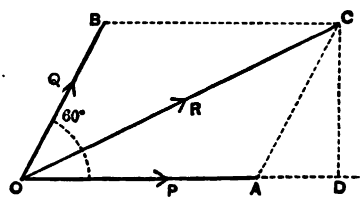

Fig. 22.

$$R^2 = \left(P + \frac{Q}{2}\right)^2 + \frac{3Q^2}{4}$$
$$= P^2 + Q^2 + PQ.$$

These are both given by the general formula by putting $\gamma = 45°$, $\cos\gamma = \frac{1}{\sqrt{2}}$ and $\gamma = 60°$, $\cos\gamma = \frac{1}{2}$ respectively.

If the two displacements be equal the resultant bisects the angle between them; for, Fig. 23, if
$$OA = AC,$$
then
$$\angle AOC = \angle ACO$$
$$= \angle BOC.$$

Join AB, cutting OC in D, then AB bisects OC at right angles. And
$$R = OC = 2OD = 2OA\cos AOC$$
$$= 2P\cos\tfrac{1}{2}\gamma.$$

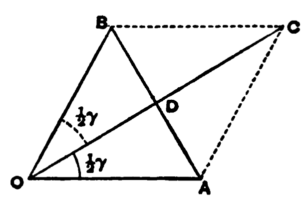

Fig. 23.

31. The Resolution of Displacements.

Just as we can combine or compound two or more displacements and find their resultant, so conversely we can resolve a single displacement into a number of others which are equivalent to it; these are called its components.

PROPOSITION 9. *To find, by a graphical construction, the components of a displacement in any two given directions.*

Let OC, Fig. 24, be the given displacement, and LM, LN the two given directions. Through O draw OA parallel to LM and through C draw AC parallel to LN. These two displacements OA, AC have OC for their resultant, hence OA, AC are components of OC and they are parallel respectively to LM and LN, that is they are drawn in the given directions.

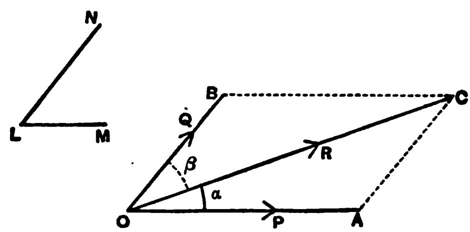

Fig. 24.

*PROPOSITION 10. *To find an expression for the components of a displacement in two given directions.*

Let OC, Fig. 24, represent R the given displacement, and let OA, AC be the components in directions making angles α, β, respectively with OC.

Then $$AOC = \alpha,$$
$$BOC = ACO = \beta.$$

Hence $$OAC = 180 - (\alpha + \beta).$$

Now in the triangle OAC the sides are proportional to the sines of the opposite angles.

Hence $$\frac{OC}{\sin OAC} = \frac{OA}{\sin ACO} = \frac{AC}{\sin AOC},$$

$$\therefore \quad \frac{R}{\sin(\alpha+\beta)} = \frac{P}{\sin\beta} = \frac{Q}{\sin\alpha}.$$

Moreover from the figure $\alpha + \beta = \gamma$.

Hence $$P = R\frac{\sin\beta}{\sin\gamma}$$

$$Q = R\frac{\sin\alpha}{\sin\gamma}.$$

PROPOSITION 11. *To find the components of a displacement in two directions at right angles.*

Let OC, Fig. 25, be the displacement R, OA, OB two directions at right angles in which the components are required.

Let $AOC = \alpha$.

Draw CA, CB perpendicular to the two directions respectively. Then OA, OB represent the components P, Q.

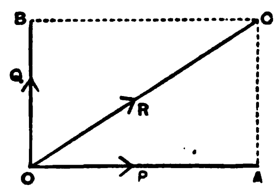

Fig. 25.

Also $\frac{OA}{OC} = \cos AOC = \cos\alpha$,

$$\therefore \quad OA = OC\cos\alpha.$$

Hence $$P = R\cos\alpha.$$

Again $\frac{OB}{OC} = \cos BOC = \sin AOC = \sin\alpha$,

$$\therefore \; OB = OC\sin\alpha.$$

Hence $Q = R\sin\alpha$.

If we put $BOC = \beta$ we have clearly

$$OB = OC\cos\beta$$

$$Q = R\cos\beta.$$

And in this case $\alpha + \beta = 90°$.

Thus, when a displacement is resolved into two others mutually at right angles, the component in each direction is found by multiplying the original displacement by the cosine of the angle between it and the direction of the component.

It must be remembered that this result is only true when the two components are at right angles.

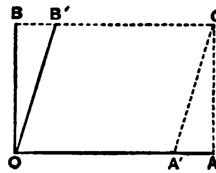

Fig. 26.

Thus, let OA, OB (Fig. 26) be two components of OC at right angles. Draw OB' making an angle γ with OA and through C draw CA' parallel to OB'. If now OC be resolved into two displacements in directions OA and OB' inclined at an angle γ, the component in the direction OA is no longer OA but OA'.

A displacement represented by OA is $R \cos \alpha$, where α is the angle between OC and OA; that represented by OA' has not this value.

32. The Composition of Velocities. In the previous propositions we have considered the composition of displacements without reference to the time taken to produce the displacement.

Now a velocity is measured by the displacement produced in the unit of time; if therefore the various lines of the figures represent displacements per second, they represent velocities and the propositions are therefore true of velocities as well as of displacements.

Thus if a point has a number of velocities communicated to it simultaneously, it will move in the direction of the resultant velocity as given by a construction similar to that of § 28 (the polygon of displacements), and its speed will be measured by the length of the line representing that resultant velocity.

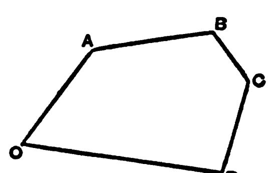

Fig. 27.

It is immaterial to the result how the particle comes to possess these various velocities; thus, a man moving across a railway carriage has his own velocity relative to the carriage, this is superposed on the velocity of the carriage along the rails, this again

on the velocity of the lines about the axis of the Earth, and this on the velocity of the Earth about the Sun; to find the resultant draw OA, Fig. 27, to represent the velocity of the man across the carriage, AB to represent the velocity of the carriage, BC that of the rails round the Earth's axis, and CD that of the Earth's centre round the Sun. Join OD, then OD is the actual velocity of the man relative to the Sun.

Or again, we may suppose the particle to be set in motion by a number of blows delivered simultaneously, one of which would give it a velocity OA, Fig. 28, a second would give a velocity OA', a third OA'', and so on, the resultant is found in the same manner; thus, from O draw OA to represent the first velocity, from A draw AB equal and parallel to OA' to represent the second, from B draw BC equal and parallel to OA'' to represent the third, and so on; then if P is the extremity of the last line thus drawn OP is the resultant velocity.

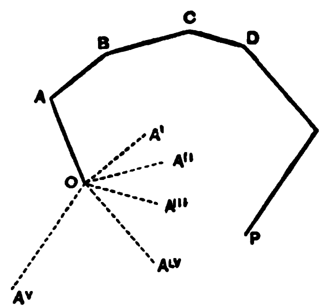

Fig. 28.

EXPERIMENT 7. *To illustrate the composition of velocities.*

As an illustration of the composition of velocities, consider the motion of a marble which is rolling with uniform speed u along a tube AB, Fig. 29, and suppose each point of the tube to be moving with uniform speed v parallel to AC. The two motions will be independent, the marble will move relatively to the tube as though the tube were at rest, while the tube moves as though the marble were not present. In AC take AL_1 equal to v, draw L_1M_1 parallel to AB, and make L_1P_1 equal to u. At the end of one second the tube will be in the position L_1M_1, the end A having come to L_1.

Since in one second the marble has moved a distance u

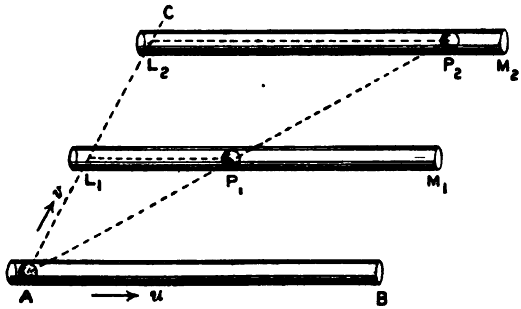

Fig. 29.

along the tube, and P_1L_1 is equal to u, the marble will be at P_1.

Again, if AL_2 is equal to $2v$, then L_2M_2 drawn parallel to AB is the position of the tube after 2 seconds, and if L_2P_2 is equal to $2u$, then P_2 is the position of the marble. Thus the position of the marble at any time can be determined, and it will be seen on making the construction that all the points thus found lie on the line AP_1 or that line produced, thus the marble moves in a straight line AP_1P_2.

Again, AP_1 is the distance traversed in one second, AP_2 the distance traversed in two seconds. Now, from the figure, AP_2 is equal to twice AP_1. Thus the distance traversed in two seconds is equal to twice that traversed in one second; continuing thus we see that the distance traversed is proportional to the time of traversing it, and hence the velocity is uniform. Again, AP_1 is the distance traversed in one second, it is therefore the resultant velocity, and AP_1 is the diagonal of a parallelogram whose sides are u and v. Thus the parallelogram of velocities is verified.

This example illustrates the method adopted in the formal proof of the proposition which is given below in a form slightly modified from the proof of the parallelogram of displacements in § 30.

33. The Parallelogram of Velocities.

Proposition 12. *If a particle possess simultaneously two velocities represented by two adjacent sides of a parallelogram, these are equivalent to a single resultant velocity represented by the diagonal of the parallelogram passing through their point of intersection.*

Let OA, OB, Fig. 30, represent the two velocities u, v respectively. Complete the parallelogram $AOBC$, and draw the diagonal OC. Then OC shall represent the resultant velocity.

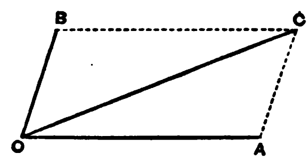

Fig. 30.

(i) Let the two velocities be uniform. Then OA, OB represent the displacements of the point in one second due to the two velocities separately. Now we may consider the motion to be made up of a displacement with velocity u along OA, and a displacement with velocity v of the line OA parallel to itself.

Owing to the first, at the end of one second, the particle would be at A, but owing to the motion of the line OA the point A at the end of one second will have come to C. Thus the particle will be at C.

Again the component velocities are uniform, i.e. the same in magnitude and direction at each instant, hence their resultant must be a uniform velocity; hence the particle has moved in one second from O to C with uniform velocity. Hence the straight line OC represents the resultant velocity.

(ii) When the two velocities are not uniform the proof given in (i) still applies, for a variable velocity can be measured by the distance which would be traversed in one second if during that second the velocity remained constant. Thus OA, OB represent the distances which would be traversed in one second by the particle moving with velocities u, v respectively, if during the second those velocities remained constant. Hence OC represents the distance which would be traversed in one second by the particle when moving with the resultant velocity, if

during the second the resultant velocity remained constant. Hence OC represents the resultant velocity.

Thus whether the component velocities be uniform or variable the diagonal OC still represents their resultant. Hence the parallelogram of velocities is true.

There is however an important distinction to be observed between the two cases. Let us suppose that O in Fig. 30 represents the original position of the particle, then *if the velocities be uniform* the position of the particle at the end of one second if it possessed the velocity u only would be A and its actual position is C. OC represents not only the velocity of the particle but also its path; it has moved with uniform velocity along the line OC from O to C.

If the velocities be variable, then OA and OB do not represent the *actual* displacements of the particle in one second due to the two velocities respectively, and therefore OC is not the *actual* displacement due to the resultant velocity. The particle at the end of a second is not at C. The line OC represents the resultant velocity but *not* the path described.

34. Composition and Resolution of Velocities.

The various propositions proved for displacements in §§ 28, 30 may now be extended to velocities. Thus we have the Triangle of Velocities (§ 30). *If two velocities be represented by two sides of a triangle* taken in order their resultant is represented by the third side taken in the reverse direction.

Hence if OAC, Fig. 31, be a triangle and if OA, AC, represent two velocities possessed simultaneously by a particle, then OC represents the resultant velocity.

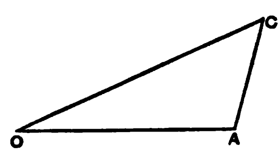

Fig. 31.

Or, putting the same result in another form. If a particle possess velocities represented in direction and magnitude by the three sides of a triangle taken in order it remains at rest.

Resultant of two velocities at right angles, § 30.

If u, v represent the two velocities and U the resultant, then

$$U = (u^2 + v^2)^{\frac{1}{2}}.$$

Resultant of two velocities inclined at an angle γ, § 30.

If u, v be the two velocities, U the resultant,

$$U = \{u^2 + v^2 + 2uv \cos \gamma\}^{\frac{1}{2}}.$$

Components at right angles of a velocity U, § 31.

Let u, v be the two components at right angles, and let u make an angle a with U, then v makes an angle $90° - a$ with U, and we have

$$u = U \cos a$$
$$v = U \cos (90° - a) = U \sin a,$$

u and v are spoken of as the *resolved parts* of the two velocities.

Components in any two directions of a velocity U, § 31.

Let u, v the two components make angles a, β, respectively with U.

$$u = U \frac{\sin \beta}{\sin (a + \beta)},$$

$$v = U \frac{\sin a}{\sin (a + \beta)}.$$

The proofs of these various propositions are identical with those given in the corresponding sections quoted, the word velocity being substituted for displacement. It is left as an exercise to the student to write them out in this form.

Examples. (1). *Find the resultant of velocities of 2 to the North, 3 to the East, 3 to the South, and 4 to the West.*

Draw a vertical line OA, Fig. 32 (a), upwards 2 cm. in length, draw BA horizontal to the right 3 cm. in length, BC vertical downwards 3 cm. in length; CD horizontal to the left 4 cm. in length. Join OD, it is the resultant required.

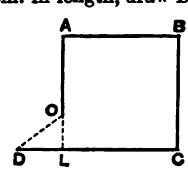

Fig. 32 (a).

Also if OL be drawn perpendicular on CD it is clear that OL is 1 cm. and LD is 1 cm.

Hence $OD = \sqrt{2}$ cm.

Thus the resultant velocity is $\sqrt{2}$ to the South-west.

Aliter. Velocities of 2 north and 3 south have clearly a resultant of 1 south, velocities of 3 east and 4 west have a resultant 1 west; the resultant of 1 south and 1 west is $\sqrt{2}$ south-west.

(2). *A boat is rowed across a river $\frac{1}{2}$ a mile wide with a velocity of 3 miles per hour, and the stream carries it down with a velocity of 4 miles per hour. Find its actual velocity and the distance parallel to the bank between the starting point and the point at which it arrives.*

The velocity of the boat is the resultant of two at right angles of 3 miles an hour and 4 miles an hour respectively, denoting it by U we have

$$U^2 = 3^2 + 4^2 = 25;$$

$$\therefore\ U = 5 \text{ miles per hour.}$$

The time taken to cross the river is independent of the motion downwards. Thus, since the river is $\frac{1}{2}$ a mile wide and the velocity at right angles to the stream is 3 miles an hour, a distance of $\frac{1}{2}$ a mile is traversed in $\frac{1}{6}$ of an hour.

Thus the time of crossing is 10 minutes.

But the stream moves at the rate of 4 miles an hour, thus in $\frac{1}{6}$ of an hour the boat is carried $\frac{2}{3}$ of a mile down.

Thus the distance parallel to the bank between the points is $\frac{2}{3}$rds of a mile.

(3). *Find the resultant of two velocities of* 50 *cm. per second and* 100 *cm. per second inclined at an angle of* 60°.

Substituting in the formula

$$U^2 = u^2 + v^2 + 2uv \cos \gamma,$$

we have

$$U^2 = 50^2 (1 + 4 + 2 \times 2 \times \tfrac{1}{2})$$
$$= 50^2 \times 7;$$
$$\therefore\ U = 50\sqrt{7} \text{ cm. per second.}$$

This might be solved as in § 30 without quoting the Trigonometrical formula.

(4). *A particle has a velocity of* 10 *cm. per second in a north-west direction, find its components to the north and to the west.*

Draw OC, Fig. 32, 10 cm. long to represent the given velocity. From O and C draw OA and CA each at 45° to OC meeting at A, then OA and AC represent the two components. Also from the figure,

$$AO = AC,$$
$$OC^2 = AC^2 + AO^2,$$
$$\therefore\ AC = \frac{10}{\sqrt{2}} = AO.$$

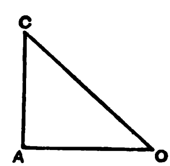

Fig. 32.

Thus each of the components is $\dfrac{10}{\sqrt{2}}$ cm. per second.

(5). *A particle has a velocity of* 15 *cm. per second which is resolved into two components at right angles. The magnitude of one component is* 9 *cm. per second, find that of the other.*

If u be the other component we have

$$u^2+9^2=15^2.$$

$$\therefore u^2=15^2-9^2=(15+9)(15-9)=24\times 6.$$

Hence $u=12$ cm. per second.

(6). *Find an expression for the resultant of a number of velocities* u_1, u_2, u_3, etc. *making angles* a_1, a_2, a_3, etc. *with a fixed line.*

Let U be the resultant and let it make an angle θ with the line.

Then the resolved parts of the resultant in any two directions at right angles must be equal to the resolved parts of the components in these two directions.

Hence resolving along and perpendicular to the fixed line

$$U\cos\theta=u_1\cos a_1+u_2\cos a_2+u_3\cos a_3+\ldots$$
$$=\Sigma\{u\cos a\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where Σ is written for abbreviation and means *the sum of a number of terms such as*

$$U\sin\theta=u_1\sin a_1+u_2\sin a_2+\ldots\ldots$$
$$=\Sigma\{u\sin a\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Hence squaring and adding, since $\sin^2\theta+\cos^2\theta=1$,

$$U^2=[\Sigma\{u\sin a\}]^2+[\Sigma\{u\cos a\}]^2$$

$$\tan\theta=\frac{\Sigma\{u\sin a\}}{\Sigma\{u\cos a\}},$$

and from these equations the resultant velocity U and its direction θ can be found.

(7). *The resultant of two velocities of* 3 *cm. per sec. and* 5 *cm. per sec. respectively is a velocity of* 7 *cm. per sec. Find the angle between the two.*

Let γ be the angle required, then if u, v be two velocities inclined at an angle γ which have a resultant U we know that

$$U^2=u^2+v^2+2uv\cos\gamma.$$

Hence
$$7^2=5^2+3^2+2\times 5\times 3\cos\gamma;$$
$$\therefore 30\cos\gamma=7^2-5^2-3^3$$
$$=49-25-9=15,$$
$$\therefore \cos\gamma=\tfrac{1}{2},$$
$$\therefore \gamma=60°.$$

Hence the angle between the component velocities is 60°.

35. Experiments on the Parallelogram Law.

The parallelogram law for the composition of displacements and velocities can be illustrated by means of the apparatus shewn in Fig. 33 and described in the following experiment.

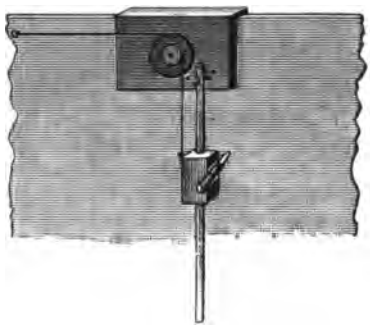

Fig. 33.

EXPERIMENT 8. *To verify the parallelogram law for the composition of velocities.*

A block of wood is made with a groove so as to slip along the horizontal edge of a drawing-board, held with its plane vertical; to this wooden block is fitted a pulley round which a string passes, one end of the string is fastened to the drawing-board and the other to the pulley; as the block is moved along the edge of the board the pulley is rotated about an axis at right angles to the board and the string is unwound. Rigidly attached to this pulley are one or more pulleys of different diameters, which revolve round the same axis as the first; one end of a string is fixed to one of these pulleys, and to the other end is attached a weight sliding on a bar which hangs vertically down from the block or can be fixed at any required angle to the horizon; the weight is thus raised or lowered as the pulley rotates and carries a clip to which a piece of chalk can be attached. If now the block be moved along the edge of the board the weight will be moved horizontally with a certain velocity, that of the block; but owing to the motion of the block the pulleys are rotated

and the string to which the weight is attached is wound up; and so motion along the bar is imparted to the weight; the actual displacement of the weight will be the resultant of these two displacements.

Perform the experiment as follows: fix the bar so as to be vertical; mark the point on the bar at which the weight is, slide the block along the edge of the board through a measured distance, and trace by means of the chalk attached to the weight the path of its motion, it will be found to be a straight line. Measure the distance along the bar through which the weight has moved and draw from the point at which the weight starts two straight lines, one horizontal and equal to the distance moved by the block, the other parallel to the bar and equal to the distance traversed by the weight along it. It will be found that the line marked by the chalk is the diagonal of the parallelogram of which the two lines are sides. Now these two lines represent the component displacements of the weight, and we see that the diagonal represents the resultant. Hence the parallelogram law is verified. Repeat the experiment for other positions of the bar carrying the weight, that is for other angles between the component velocities. The ratio of the displacements in the two directions depends on the diameters of the pulleys used and can be varied by using different sized pulleys.

*36. Relative Velocity.

It has already been pointed out that all motion with which we are concerned is relative motion, and we have seen that a particle A is in motion relative to a second particle B, when the length or direction of the line AB varies. It is often desirable to determine the motion of one particle relative to a second which is itself in motion. Now it is clear that the relative motion of two particles is not altered by superposing on both the same velocity; for example, the relative motion of two flies crawling on the window of a railway carriage is the same, whether the carriage be at rest or in motion.

We can apply this then to find the motion of A relative to B thus. Superpose on the motions of A and B a velocity equal

and opposite to that of B. The relative motion is unaltered, the particle B is reduced to rest while A moves with a velocity which is the resultant of its own velocity and the reversed velocity of B. This resultant motion is now the motion of A relative to B.

Example. *The paths of two ships intersect at right angles, one ship, moving with a velocity of 13 miles an hour, is 15 miles from the point of intersection, the other, moving with a velocity of 20 miles per hour, is 10 miles from this point; find the least distance between the ships.*

Let O, Fig. 34, be the point of intersection of the paths, A the position of the first ship 15 miles from O, B that of the second 10 miles from O. Bisect OA in C and join BC, from A draw AD perpendicular to BC produced.

Then $\quad BO=10$ miles,

$OC=CA=7\cdot5$ miles,

$BC=12\cdot5$ miles,

for $\quad BC^2=BO^2+OC^2$.

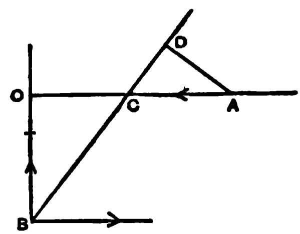

Fig. 34.

If we take BO to represent a velocity of 20 miles an hour, OC will represent one of 15 miles per hour in a direction opposite to that in which A is moving. If then we superpose on the ships a velocity represented by OC, the ship A will be reduced to rest while B will move in the direction BC with a velocity represented on the same scale by BC; this is a velocity of 25 miles an hour. Thus the relative motion of the two ships is represented by a velocity of 25 miles an hour in the direction BC.

The two ships will be nearest apart when B has arrived at D, and this least distance is given by AD.

Now
$$\frac{AD}{AC}=\frac{OB}{BC};$$

$$\therefore\ AD=\frac{AC\,.\,OB}{BC}=\frac{7\cdot5\times10}{12\cdot5}$$

$$=6 \text{ miles}.$$

Thus the least distance apart of the ships will be 6 miles.

Moreover
$$\frac{CD}{AD}=\frac{CO}{OB};$$

whence
$$CD=4\cdot5 \text{ miles},$$

$$\therefore\ BD=17 \text{ miles}.$$

Hence since the relative velocity of B along BD is 25 miles per hour, the ship B will reach D in $\frac{17}{25}$ hour from the time at which it was at B, and at this time the two ships will be the closest together.

*37. Angular Velocity.

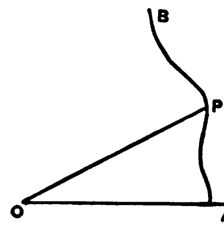
Fig. 35.

Let P, Fig. 35, be a point which is moving along a plane curve APB, and let O be any fixed point in the plane of the curve and OA a fixed line through O. As P moves the angle POA varies; the rate of change of this angle is called the angular velocity of the point P about O, and is measured in general by the ratio of the change in the angle to the interval of time during which that change has occurred when that interval is made sufficiently small.

When the angular velocity is uniform it is measured by the ratio of the angle θ, described in the interval of time t seconds, to the time, so that in this case if ω be the uniform angular velocity we have

$$\omega = \frac{\theta}{t},$$

$$\theta = \omega t.$$

***38. Motion with uniform speed in a circle.** If the curve described be a circle, with O, Fig. 36, as centre, we can find a relation between the uniform angular velocity about O and v the uniform speed of the particle in the circle.

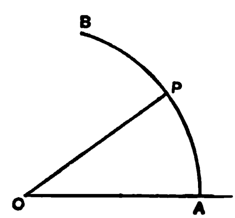
Fig. 36.

For if s be the arc described in time t measured from A we have, since the speed is constant,

$$s = vt.$$

But if a be the radius of the circle, and θ the circular measure of the angle AOP, then

$$\theta = \frac{s}{a}.$$

But $$\theta = \omega t,$$

$$\therefore \quad \omega t = \frac{vt}{a}.$$

Hence $$v = a\omega.$$

Hence the speed in the circle is found by multiplying the angular velocity by the radius of the circle.

Example. *Assuming the earth to be a sphere whose radius is* $6{\cdot}436 \times 10^6$ *metres, find in metres per second the velocity of a point on the equator.*

The earth rotates uniformly through an angle whose circular measure is 2π (44/7) in 24 hours.

$\therefore$ its angular velocity is $\dfrac{44}{7 \times 24 \times 3600}$.

Hence the velocity of a point on the equator is

$$\frac{44 \times 6436 \times 10^3}{7 \times 24 \times 3600},$$

and this reduces to 468 metres per second.

A relation identical with the above holds between the speed, the radius of the circle and the angular velocity about the centre even when the two are not uniform, provided that v and ω stand for the values of the speed and the angular velocity at the same moment of time.

39. Graphical Representation of Space passed over by a particle. Draw a horizontal line OX, Fig. 37, divide it into a number of equal parts in the points N_1, N_2, N_3 etc. and let each such part represent a small interval of time. From each point draw lines P_1N_1, P_2N_2, etc. at right angles to OX to represent the velocity of the particle at the end of the corresponding interval; join the points P_1P_2.... If the intervals be sufficiently small the line joining these points will be a continuous curve. Such a curve is called a velocity curve; it is defined by the property that if a perpendicular PN be drawn from any point on it to meet the time line OX in N then PN is the velocity of the particle at the time represented by ON.

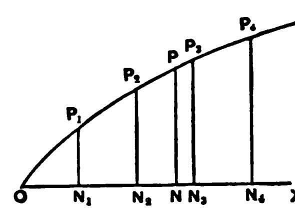

Fig. 37.

Let us now suppose the velocity to be constant; the lines P_1N_1, P_2N_2, etc. will all be of the same length, the velocity curve is a straight line such as PP' parallel to OX, Fig. 38. Let P, P' be two points on the curve and $PN, P'N'$ perpendicular to OX, let t be the time represented by NN' and let v be the constant velocity, s the distance traversed.

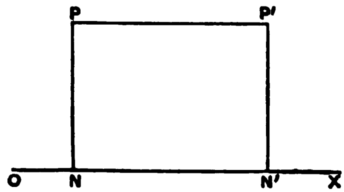

Fig. 38.

Then $$v = PN,$$
$$t = NN'.$$

Now $$s = vt,$$
$$\therefore\ s = PN \times NN' = \text{area } PNN'P'.$$

Thus in this case of uniform velocity *the area between the velocity curve, the line* OX *and two lines perpendicular to the line* OX *represents graphically the space traversed by the particle.*

Some further consideration shews us that this proposition is always true whether the velocity be uniform or variable.

For we have seen that we may approach the case of a *continuously* varying velocity by dividing the time up into a large number of small intervals and supposing the velocity to remain constant during each interval but to change suddenly at the end of every interval.

Draw the velocity curve for such a case supposing for the present the intervals during which the velocity is constant to be *seconds*. It will consist, as shewn in Fig. 39, of the series of horizontal and vertical straight lines

$$P_1R_1P_2R_2P_3R_3 \ldots$$

alternately parallel and perpendicular to OX. During the time N_1N_2

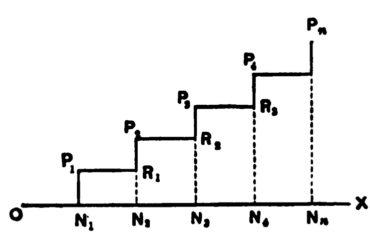

Fig. 39.

the velocity is constant and equal to P_1N_1; the space traversed is the area $P_1N_1N_2R_1$; at the time N_2 the velocity changes to P_2N_2 increasing by P_2R_1; for the next second N_2N_3 the velocity is constant and equal to P_2N_2, the space is represented by $P_2N_2N_3R_2$, and so on. Thus in this case the whole space traversed is the area between the velocity curve, the line OX and the two bounding lines P_1N_1 and P_nN_n.

This result will be equally true if we divide each second into a large number of parts and suppose the velocity to change at the end of each part. Instead of a single step between P_1 and P_2 we obtain a large number of steps; instead of a single parallelogram such as $P_1N_1N_2R_1$ we have a large number; the sum of the areas of these parallelograms is still the space traversed. If now we make the number of parts sufficient each individual step will be indefinitely small, the broken line will merge into the continuous velocity curve, and the sum of the parallelograms will become the area of that curve. Thus the space traversed during a given time is given by the area bounded by the velocity curve, the line OX and two lines perpendicular to OX drawn from points on OX which represent the beginning and the end of the time. Whenever then we can calculate the area of this curve, we can find the space traversed by the particle. In the important case considered in the next chapter the velocity increases uniformly with the time and the curve is a straight line. The area is bounded by straight lines and can therefore be easily calculated.

By drawing a diagram to scale on squared paper and then determining the area by the method given in Experiment 3, we can find the space traversed in many cases in which we are given the relation between the velocity and the time. In solving such a question it is necessary to be careful as to the units in which the lines in the diagram are measured. Suppose, for example, 1 cm. along the time line represents an interval of 1 second, and 1 cm. at right angles to this a velocity of 1 cm. per second; then an area of 1 sq. cm. represents the distance traversed in 1 second by a particle moving with a velocity of 1 cm. per second; that is, it represents a line 1 cm. in length; if however we had taken a length of 1 cm. at right angles to the time line to represent a velocity of v cm. per second, then an area of 1 sq. centimetre would represent a length of v centimetres.

Examples.

(1). The velocities at the ends of 1, 2...10 seconds are 5, 7, 9...28 cm. per second, find by a diagram the space traversed in 10 seconds.

(2). The velocities at the ends of 1, 2...10 seconds are $1^2, 2^2 \ldots 10^2$ cm. per second, find by a diagram the space traversed in 10 seconds.

These examples are left for the student to solve with the aid of a ruler and squared paper.

EXAMPLES.

UNIFORM SPEED.

1. Find in feet per second the following velocities:

(1) 10 miles per hour; (2) a quarter of a mile in 44 seconds; (3) 92000000 miles in $8\frac{1}{4}$ minutes; (4) 25000 miles in 24 hours.

2. Find in centimetres per second the velocity of a body which traverses

(1) a cm. in b seconds; (2) a circle of 10 cm. radius in 1 second; (3) 76 cm. in 10 minutes; (4) the perimeter of a square 1 foot in edge in 1 minute.

3. The speed of a steamer is 22 knots, reduce this to cm. per second.

4. A particle has a velocity of 30 miles per hour, how many feet does it traverse (1) in 1 minute, (2) in a day, (3) in a year?

5. A man walks a mile in 10 minutes, a second mile in 12 and a third in 15; he runs a fourth mile in 5 minutes; find his average speed (1) in feet per second, (2) in miles per hour.

6. A and B start to walk towards each other from two places 6 miles apart. A walks twice as fast as B. Where will they meet? The meeting takes place 50 minutes after the start, find the speed of each.

7. A starts along a road at a speed of 3 miles an hour, after 40 minutes B follows at a speed of 5 miles an hour, how far must B go before overtaking A?

8. The velocity of sound is 1100 feet per second, a man in front of a cliff claps his hands and hears an echo after 5 seconds, how far is he from the cliff?

9. A man climbs a hill inclined at 30° to the horizon, if he rises vertically 1000 feet in an hour find his speed in feet per second.

10. The radius of the Earth's Orbit is 92 million miles and the radius of the Earth 4000 miles, compare the velocities of a point on the equator at midday and at midnight.

11. Find the resultants of the following pairs of velocities in directions at right angles to each other; the velocities are all expressed in centimetres per second:

(1) 3 and 4; (2) 6 and 8; (3) 12 and 15; (4) v_1 and v_2, where $v_1+v_2=7$, $v_1-v_2=1$.

12. Find by a graphical construction and by the formulæ the resultants of the following velocities:

(1) 3 and 4 at 60°; (2) 6 and 8 at 45°; (3) 1 and 2 at 30°; (4) 1 and 2 at 60°.

13. A boat is rowed across a river at the rate of 3 miles per hour, the river is flowing at the rate of 4 miles per hour; find the velocity of the boat.

14. A ship is sailing at the rate of 10 miles an hour and a sailor climbs the mast 200 feet high in 30 seconds. Find his velocity relative to the Earth.

15. The paths of two ships steaming North and East respectively, with velocities of 12 and 16 miles per hour, meet. The two ships are each 12 miles distant from the point of intersection. Determine after what time they will be closest together and what that closest distance will be.

16. Two equal velocities have a resultant equal to either, shew that they are inclined to each other at 120°.

17. The resultant of two velocities u and v is equal to u, and its direction is at right angles to that of u. Shew that v is equal to $u\sqrt{2}$.

18. Find by a graphical construction or otherwise the resultant of the following velocities in the directions of the sides of a square taken in order:

(1) 1, 2, 2, 1; (2) 3, 4, 5, 6; (3) 2, 5, 6, 3; (4) 7, 8, 4, 5.

19. Find by a graphical construction or otherwise the resultant of the following velocities in the directions of the sides of an equilateral triangle taken in order:

(1) 3, 3, 3; (2) 4, 5, 6; (3) 5, 8, 10; (4) 6, 9, 12.

20. Find the horizontal and vertical components of the following velocities:

(1) 1000 ft. per second in directions inclined respectively at 30°, 45° and 60° to the horizon.

(2) 25 miles per hour at 60° to the vertical.

21. Resolve a velocity of 1000 feet per second into two equal velocities inclined at 60° to each other.

22. A velocity of 500 feet per second is resolved into two at right angles, one of these is 250 feet per second, find the other.

23. A velocity of 5 miles an hour to the East is changed into one of 5 miles an hour to the North; find the change in velocity.

24. A velocity represented by one side AB of an equilateral triangle ABC becomes changed into one represented by the side AC; find the change in velocity.

25. Find by a graphical construction or otherwise the resultant of the following velocities, which are given in centimetres per second:

(*a*) 15 to the North, 20 to the East, $20\sqrt{2}$ to the North-west, 35 to the West.

(*b*) 1, 2, 3, 4, 5, 6 parallel respectively to the sides of a regular hexagon.

26. One of the rectangular components of a velocity of 60 miles per hour is a velocity of 30 miles per hour; find the other component.

27. A body moves during each of 5 consecutive minutes with velocities respectively of 1, 2, 3, 4, 5 feet per second; find the space traversed.

28. The spaces traversed up to the end of 1, 2, 3 and 4 minutes by a body moving with constant velocity during each minute are 2, 8, 18 and 32 feet respectively. Shew on a diagram the velocity during each minute.

29. The velocity of a body starting from rest increases by 1 foot per second at the end of every second of its motion. Determine by means of a diagram or otherwise the space passed over in t seconds.

30. The components in two directions of a velocity of 30 miles per hour are velocities of 15 and 25 miles per hour, determine their directions.

31. Two velocities u and v have a resultant U which makes an angle α with the direction of u; if u be increased by U while v is unchanged shew that the new resultant makes an angle $\frac{\alpha}{2}$ with the direction of u.

32. Two particles are projected simultaneously with equal velocities from the points A and B, one from A towards B, and the other in a direction at right angles to AB; find how far the former will have travelled towards B when the two particles are nearest to one another.

33. If a point begins to move with velocity u, and at equal intervals of time τ, a velocity v is communicated to it; find the space described in n such intervals.

34. Compare the velocities of two trains, one travelling with a velocity of 50 miles per hour and the other with a velocity of 55 feet per second.

CHAPTER III.

KINEMATICS. ACCELERATION.

40. Change of Velocity. The velocity of a particle may change either in magnitude or in direction or in both these respects.

Let OA represent the velocity of a particle at a given instant; if the velocity remain uniform, OA will continue to represent it; suppose however that the velocity changes and that after an interval it is represented by OB. If the change occur in the magnitude only, the particle will continue to move in the same direction as before.

OAB will be a straight line, and AB, Fig. 40, will represent the velocity which must be added to the original velocity OA to give OB.

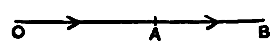

Fig. 40.

If the change in velocity take place in direction as well as in magnitude, OA and OB will be inclined to each other, Fig. 41, but AB will still represent the change in the velocity, for by the parallelogram of velocities AB is the velocity which must be compounded with OA to give OB.

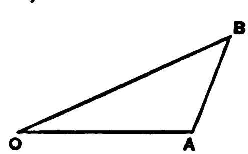

Fig. 41.

Thus AB represents in direction and magnitude the velocity which must be superposed on OA to change it into OB.

If then we represent by two straight lines drawn from a point the initial and final velocities of a particle, the line joining the extremities of these two lines represents the **Change of velocity** of the particle.

Examples. (1). *A particle moving North-east with a velocity of* 1 *foot per second is observed, after a time, to be moving East with a velocity of* $\sqrt{2}$ *feet per second, find the change in velocity.*

Here, Fig. 42,

$OA=1$, $OB=\sqrt{2}$ and $AOB=45°$.

Draw AC normal to OA meeting OB in C; then $ACB=45°$ and $AC=AO=1$.

Hence $OC^2=2=OB^2$, thus C and B coincide, and AB the velocity added is 1 ft. per second in the South-east direction.

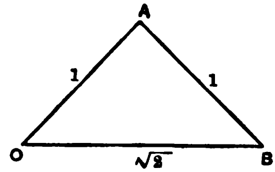

Fig. 42.

(2). *A velocity of* 10 *feet per second is changed into one of* 10 *feet per second inclined at* 60° *to the former, find the change in velocity.*

In this case,

$OA=OB=10$ and $AOB=60$,

$\therefore\ \angle OAB=\angle OBA=\frac{1}{2}(180-60)=60°$.

Thus $AB=OA=10$.

The additional velocity is one of 10 feet per second inclined at 60° to OA.

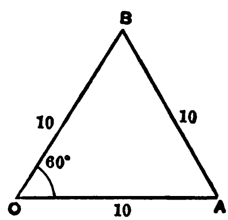

Fig. 43.

41. Acceleration. When the velocity of a particle is variable it is said to have acceleration.

DEFINITION. *The* **Acceleration** *of a particle is its rate of change of velocity.*

Acceleration may be **Uniform** or **Variable.** Uniform acceleration is measured by the ratio of the change of velocity to the interval of time during which that change has occurred, i.e. by the change of velocity in one second; variable accelera-

tion is measured by the same ratio when the interval is sufficiently small, that is, by the change in velocity which would take place in one second if during that second the velocity changed uniformly.

The numerical measure of an acceleration is the number of units of velocity added per second. Now velocity is measured by the number of units of space traversed per second.

When, then, we state that the acceleration of a particle moving with uniform acceleration is a, we mean that in each second an additional velocity of a cm. per second is given to the particle. To define then an acceleration we must know the number of units of space per unit time which are added to the velocity, and further we must remember that this additional velocity is conferred in the unit of time.

Just then as when considering a velocity we speak of so many centimetres per second, so when dealing with acceleration we speak of so many centimetres per second per second.

DEFINITION OF UNIT ACCELERATION. *A particle has* **Unit Acceleration** *when its velocity increases in each second by* 1 *centimetre per second.*

If the units of space or time be changed the numerical measure of a given acceleration is changed also.

The method of calculating these changes is shewn below.

Example. *A particle has an acceleration of* 32·2 *feet per second per second, find its value* (*a*) *in cm. per sec. per sec.,* (*b*) *in yds. per min. per min.*

For (*a*) we have 1 ft. = 30·48 cm.

Now in 1 sec. a vel. of 32·2 ft. per sec. is added,

∴ in 1 sec. a vel. of $32 \cdot 2 \times 30 \cdot 48$ cm. per sec. is added.

∴ the new measure is $32 \cdot 2 \times 30 \cdot 48$ cm. per sec. per sec.

This reduces to 981·5 cm. per sec. per sec.

For (*b*), 1 ft. = $\frac{1}{3}$ yd., 1 min. = 60 sec.

In 1 sec. a vel. of 32·2 ft. per sec. is added,

$$\therefore \text{ in 1 sec. a vel. of } \frac{32{\cdot}2}{3} \text{ yds. per sec. is added,}$$

$$\therefore \text{ in 1 sec. a vel. of } \frac{32{\cdot}2 \times 60}{3} \text{ yds. per min. is added,}$$

$$\therefore \text{ in 1 min. a vel. of } \frac{32{\cdot}2 \times 60 \times 60}{3} \text{ yds. per min. is added.}$$

Thus the new measure is $\dfrac{32{\cdot}2 \times 60 \times 60}{3}$ yds. per min. per min.

This reduces to 38640 yds. per min. per min.

It will be noticed that in (*b*) the change in the unit of time comes in twice. The reason for this is clear, the unit of time affects the measure of the velocity and affects also the time during which, when calculating the acceleration, the change in the velocity is to be reckoned.

42. Uniform acceleration in the direction of motion. The change in the velocity of a body may be a change in magnitude, in direction or in both.

For the present we deal only with the case of a body moving in a straight line with uniform acceleration.

The change of velocity will be one of magnitude only, and that change will be a uniform one, the speed will vary but not the direction of motion. The velocity may either increase or decrease; in the first case the acceleration is positive, in the second negative.

PROPOSITION 13. *To determine the velocity of a body moving in a straight line with uniform acceleration in terms of the initial velocity, the acceleration and the time of motion.*

Let the initial velocity be u, the velocity after t seconds v, and the acceleration a.

In 1 second a velocity of a centimetres per second is added and the acceleration is uniform.

Hence in $2''$ the velocity added is $2a$,

in $3''$ the velocity added is $3a$,

and in t'' the velocity added is at.

Thus at the end of t seconds the velocity is

$$u + at.$$

Hence $$v = u + at.$$

If the velocity decreases with the time, the acceleration is negative and we have

$$v = u - at.$$

The proposition can be put rather differently thus.

The change in velocity in t'' is $v - u$. Therefore the change per second is $(v - u)/t$.

But the acceleration is the change of velocity per second.

Hence $$a = \frac{v-u}{t},$$

$$\therefore\ v - u = at, \text{ or } v = u + at.$$

PROPOSITION 14. *To draw*[1] *the velocity curve for a particle moving with uniform acceleration.*

Draw a horizontal line OX, Fig. 44, to represent time and a vertical line OY to represent velocity. Choose a convenient length to represent the unit of time, and also a convenient length to represent the unit of velocity.

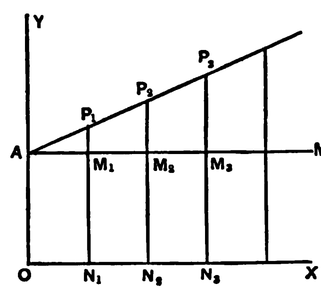

Fig. 44.

Mark off along OY a length OA to represent the initial velocity u.

Through A draw AM parallel to the time line and, commencing from A, divide AM in M_1, M_2, M_3, etc. into equal parts, each of which shall represent 1 second. At M_1, M_2, etc. draw P_1M_1, P_2M_2, P_3M_3, etc. perpendicular to AM, and produce these to meet the time line OX in N_1, N_2, etc. Make P_1M_1 equal to a, P_2M_2 equal to $2a$, P_3M_3 equal to $3a$, etc. Then these

[1] For this and similar purposes squared paper such as is used in Experiment 3 is convenient.

various lines represent the increments of the velocity up to the end of the first, second, third, etc. second, and the lines P_1N_1, P_2N_2, P_3N_3, etc. represent the actual velocity at the end of the first, second, third, etc. second.

Thus the line AP_1P_2 ... represents the velocity curve and the construction shews that it is a straight line.

Examples. (1). *A particle moving from rest with uniform acceleration has a velocity of* 160 *ft. per second after* 5 *seconds, find its acceleration.*

In each second a velocity of 160/5 feet per second is produced.
Hence the acceleration $=\frac{160}{5}=32$ ft. per sec. per sec.

(2). *A particle moving under a negative acceleration of* 32 *feet per second per second is projected with a velocity of* 160 *feet per second. Find when it will come to rest and what will be the velocity after* 10 *seconds.*

In each second a velocity of 32 feet per second is destroyed.
Therefore the initial velocity of 160 feet per second will be destroyed in 160/32 seconds.
Thus the particle is instantaneously at rest after 5 seconds.
The acceleration now produces in each second a velocity in the opposite direction of 32 feet per second. Therefore after 5 seconds more, i.e. at the end of 10 seconds, the velocity will be

$$5\times(-32) \text{ or } -160 \text{ feet per second.}$$

Aliter. Let v be the velocity after t seconds.

Then $$v=160-32\,.\,t.$$

If t_1 represent the time at which the particle is at rest, at which therefore v is zero, we have $O=160-32t_1$;

$$\therefore\ t_1=\tfrac{160}{32}=5 \text{ seconds.}$$

Again after 10 seconds,

$$v=160-32\times10=-160 \text{ ft. per sec.}$$

(3). *Draw the velocity curve in the case of* (2).

Draw the time and velocity lines OX and OY, Fig. 45. In OY take OA to represent a velocity of 160 ft. per second. Draw a line from A parallel to OX and in it take M_1 so that AM_1 may represent 1 second; from M_1 draw M_1P_1 vertically down to represent a velocity of 32 feet per second. Join AP_1 and produce it, AP_1 is the required velocity curve. It meets the line OX in N, where from the figure,

$$ON=5AM_1.$$

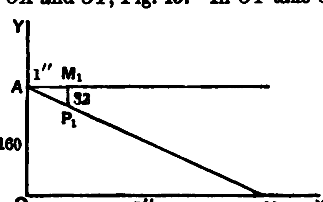

Fig. 45.

Hence ON represents 5 seconds.

43. Acceleration, space traversed and time of motion.

PROPOSITION 15. *To find the space passed over in a given time by a body starting from rest and moving with uniform acceleration.*

The space passed over is given by the area of the velocity curve which, in this case, will be a straight line passing through the point O from which the time and velocity lines are drawn. Let ON, Fig. 46, represent the time t and NP perpendicular to ON the velocity at the end of the time interval.

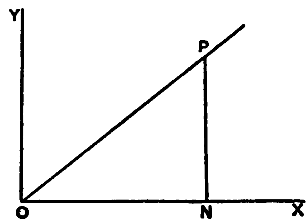

Fig. 46.

Then $\qquad PN = at.$

Join OP; the velocity curve is the line OP and the space s required is the area of the triangle OPN.

Now the area of a triangle is half the product of the base and the altitude;

$$\therefore\ s = \text{area } OPN = \tfrac{1}{2}\, PN\,.\,ON$$
$$= \tfrac{1}{2}\, at\,.\,t = \tfrac{1}{2}\, at^2.$$

Hence $\qquad s = \frac{1}{2}\, at^2.$

Thus the space passed over in the first second is $\frac{1}{2}\, a$ while the velocity at the end of that second is a. *The space traversed is found by multiplying half the acceleration by the square of the time.*

PROPOSITION 16. *To find the space passed over by a particle moving with uniform acceleration when the particle starts with an initial velocity.*

The space required will be the area of the velocity curve. In OY, Fig. 47, take OA equal to the initial velocity u; let ON represent the time t, and NP perpendicular to OX, the velocity after t seconds,

so that $\qquad PN = u + at.$

Join AP. Then AP will be the velocity curve and the space required is the area $OAPN$.

Draw AM parallel to OX to meet PN in M.

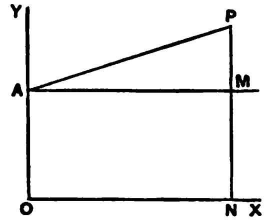

Fig. 47.

Then $PM = at$, $MN = OA = u$.

Hence

$$\begin{aligned} s &= \text{area } OAPN \\ &= \text{parallelogram } OAMN \\ &\qquad + \text{triangle } APM \\ &= OA \times ON + \tfrac{1}{2} PM \times AM, \end{aligned}$$

and $$AM = ON = t.$$

Hence $$\begin{aligned} s &= ut + \tfrac{1}{2} at \, . \, t \\ &= ut + \tfrac{1}{2} at^2. \end{aligned}$$

Thus the space actually traversed is found by adding together the spaces the particle would traverse (1) if it moved with the constant velocity u and (2) if it started from rest with the constant acceleration a.

PROPOSITION 17. *To find the average velocity of a particle moving with uniform acceleration.*

We can put the last formula in a different form thus: let v be the final velocity of the particle, then $v = PN$.

Join AN, then the figure $OAPN$ is made up of the two triangles OAN and PAN; the bases of these triangles are OA and PN and their altitude is ON.

Thus

$$\begin{aligned} s &= \text{area } OAPN = \text{triangle } OAN + \text{triangle } APN \\ &= \tfrac{1}{2} OA \, . \, ON + \tfrac{1}{2} PN \, . \, ON \\ &= \tfrac{1}{2} (OA + PN) \, ON \\ &= \tfrac{1}{2} (u + v) \, t. \end{aligned}$$

Now we know, § 24, that when a particle moves with variable speed the space traversed is found by multiplying the average speed and the time.

In this case, therefore, the average speed so defined is $\frac{1}{2}(u+v)$.

Thus, *the average speed of a particle moving with uniform acceleration is half the sum of the initial and final speeds.*

The above formula may be applied to the case in which a is negative, we have then

$$s = ut - \tfrac{1}{2}at^2$$
$$= \tfrac{1}{2}(u+v)t,$$

for $$v = u - at.$$

In a later chapter (§ 68) experiments will be given by which the student can verify for himself the truth of the formulæ proved in this section. Those who have a difficulty in following the steps of the proof may adopt the experimental proof. The argument just given may be made clearer to some by giving the following algebraical proof which sums up in symbols its results.

Let us suppose the whole time t divided up into a series of n equal small intervals each equal to τ, so that $n\tau = t$.

At the beginning of each interval the velocity will have the values respectively u, $u+a\tau$, $u+2a\tau, \ldots u+\overline{n-1}\,a\tau$, and at the end of each interval it will have the values

$$u+a\tau, \quad u+2a\tau, \quad u+3a\tau, \ldots u+na\tau.$$

The space traversed will be greater than that which would be traversed, if during each interval the particle moved with the velocity which it has at the beginning of the interval, and less than that which would be traversed if during each interval the particle moved with the velocity it has at the end of the interval.

In the first case the space would be s_1, where we have

$$s_1 = u\tau + (u+a\tau)\tau + (u+2a\tau)\tau + \ldots\ldots + (u+\overline{n-1}\,a)\tau;$$
$$\therefore\ s_1 = un\tau + a\tau^2(1+2+\ldots n-1)$$
$$= un\tau + a\tau^2\frac{n(n-1)}{2}.$$

Now $$\tau = \frac{t}{n};$$

$$\therefore\ s_1 = ut + \frac{1}{2}at^2\left(1-\frac{1}{n}\right),$$

and in the second case the space would be s_2, where

$$s_2 = (u + a\tau)\,\tau + (u + 2a\tau)\,\tau + \ldots\ldots + (u + na\tau)\,\tau$$
$$= un\tau + a\tau^2(1 + 2 + 3 + \ldots + n)$$
$$= ut + \frac{1}{2}at^2\left(1 + \frac{1}{n}\right).$$

But s lies between s_1 and s_2 and these two quantities can both be made as nearly equal as we please to $ut + \frac{1}{2}at^2$ by making n very large, for then $1/n$ vanishes.

Hence $$s = ut + \tfrac{1}{2}at^2.$$

44. Acceleration, Velocity and Space traversed.

Proposition 18. *To find a relation between the velocity, the acceleration, and the space traversed for a particle moving with uniform acceleration.*

Let u be the initial velocity, v the velocity after t seconds during which the particle has traversed a distance s and a the acceleration.

We have proved that

$$v - u = at,$$
$$s = ut + \tfrac{1}{2}at^2.$$

We wish to eliminate t from these equations.

The first gives us

$$t = \frac{v - u}{a}.$$

Hence $$s = \frac{u(v - u)}{a} + \tfrac{1}{2}a\frac{(v - u)^2}{a^2},$$

$$\therefore\ 2as = 2uv - 2u^2 + v^2 + u^2 - 2uv$$
$$= v^2 - u^2.$$

Hence $$v^2 = u^2 + 2as.$$

If the acceleration be negative we start with

$$v - u = -at,$$
$$s = ut - \tfrac{1}{2}at^2,$$

and find $$v^2 - u^2 = -2as.$$

We can put the proof otherwise, thus

we have $$a=\frac{v-u}{t},$$

$$s=\tfrac{1}{2}(v+u)t.$$

Hence on multiplication,

$$as=\tfrac{1}{2}(v-u)(v+u)=\tfrac{1}{2}(v^2-u^2),$$

or $$v^2=u^2+2as.$$

45. Formulæ connected with uniform acceleration. We have thus proved the following formulæ in which the symbols have the meanings attached to them in the preceding sections.

$$v=u+at \quad\text{.........................(i)},$$

$$s=ut+\tfrac{1}{2}at^2 \quad\text{.........................(ii)},$$

$$v^2=u^2+2as \quad\text{.........................(iii)}.$$

We may also write (ii) as

$$s=\tfrac{1}{2}(v+u)t \quad\text{....................(iv)}.$$

If the particle start from rest u is zero, and the equations become

$$v=at \quad\text{.........................(i) } a,$$

$$s=\tfrac{1}{2}at^2 \quad\text{.........................(ii) } a,$$

$$v^2=2as \quad\text{.........................(iii) } a,$$

$$s=\tfrac{1}{2}vt \quad\text{.........................(iv) } a.$$

Examples. (1). *A particle starts with a velocity of* 3 *cm. per sec. and an acceleration of* 2 *cm. per sec. per sec., find its velocity after* 10 *sec. and the distance traversed in* 10 *sec.*

Let v be the velocity after 10 seconds, s the distance traversed.

Then $$v=3+2\times 10=23 \text{ cm. per sec.}$$

$$s=3\times 10+\tfrac{1}{2}\cdot 2\cdot 10^2=130 \text{ cm.}$$

(2). *How far must the particle, moving as in Example* (1), *move in order that its velocity may become* 5 *cm. per second?*

To solve this we need equation iii, giving a relation between the velocity and the space.

If the space required be s cm., we have

$$5^2=3^2+2\times 2\times s,$$

$$\therefore\ 4s=5^2-3^2=16;$$

$$s=4 \text{ cm.}$$

(3). *A particle has a velocity of* 20 *cm. per second and an acceleration of* -5 *cm. per sec. per sec., how far will it move before coming to rest?*

If v be the velocity after it has traversed s cm. we have

$$v^2 = 20^2 - 2 \times 5 \times s.$$

If the particle is at rest for a moment, we have v zero, and hence

$$10s = 20^2 = 400;$$

$$\therefore s = 40 \text{ cm.}$$

Thus the particle is brought to rest after moving 40 cm.; if the acceleration continue to act it will only remain at rest for an instant, and then commence to retrace its path passing through the starting point with its initial velocity.

(4). *A particle starts with a velocity* u *and an acceleration* $-$a; *shew that it comes to rest after an interval* u/a *seconds and passes through the starting point again after an interval* 2u/a *seconds.*

The velocity after t seconds is $u - at$; when the particle is at rest this is zero,

$$\therefore u - at = 0;$$

$$\therefore t = \frac{u}{a}.$$

The distance of the particle from the starting point at t'' is

$$ut - \tfrac{1}{2}at^2.$$

When the particle is at the starting point this distance is zero. Then

$$ut - \tfrac{1}{2}at^2 = 0, \quad \therefore t = 0;$$

or $$u - \tfrac{1}{2}at = 0;$$

whence $$t = \frac{2u}{a}.$$

Thus the particle is at the starting point initially and reaches it again after an interval $2u/a$.

During half this interval the particle is moving from the starting point, during the second half it is moving to it.

(5). *A particle has an initial velocity of* 125 *cm. per sec. and an acceleration of* (*a*) 10 *cm. per sec. per sec.,* (*b*) -10 *cm. per sec. per sec. How long will it take in either case to move over* 420 *cm.?*

We know the initial velocity, the distance traversed and the acceleration and require to find the time.

This is given us by equation ii. Let it be t seconds, then for (a),

$$420 = 125t + \tfrac{1}{2}10t^2,$$

$$\therefore t^2 + 25t - 84 = 0;$$

solving the quadratic

$$t=\frac{-25\pm\sqrt{625+336}}{2}$$

$$=\frac{-25\pm31}{2}=3 \text{ or } -28.$$

From the solution $t=3$ we see that, 3 seconds after starting, the particle will be at a distance of 420 cm. from the starting point. Now as to the solution $t=-28''$, we infer from this that it is possible to start the particle from a position 420 cm. from the starting point with such a velocity that after 28 seconds it is at the starting point and is moving with a velocity of 125 cm. per second. Let O, Fig. 48, be the original starting point, A a point 420 cm. to the right of O; then if the particle is projected towards A with a velocity of 125 cm. per second it will arrive at A in 3 seconds, this is the first solution. But it is also possible to start the particle from A towards O with such a velocity that it passes through O to B, comes to rest for an instant at B and then returns to O, arriving at O with a velocity of 125 cm. per second, 28 seconds after it has left A; if this be possible then we may say that the particle was at A -28 seconds before leaving O.

Fig. 48.

And this is clearly possible. In the first case, the particle arrives at O with a velocity of 155 cm. per second, viz. its original velocity of 125 cm. per sec. and the velocity of 30 cm. per sec. generated in $3''$ by the acceleration. Suppose now it be projected from A towards O with this velocity. It will arrive at O after 3 seconds and have a velocity of 125 cm. per second; it will then continue to move towards B for 12·5 seconds, in which time the velocity of 125 cm. per sec. will be destroyed. Thus it arrives at B 15·5 secs. after leaving A. It will then return from B to O and will arrive at O after another 12·5 secs. with the velocity of 125 cm. per second. Thus the interval between the start at A and the time at which the particle reaches O with a velocity of 125 cm. per sec. towards A is $15·5+12·5$ or 28 seconds. Thus under the given conditions of acceleration and velocity the particle might be at A, 420 cm. from O, either 3 seconds after passing O, or 28 seconds before passing O.

(*b*) Taking now the other case in which the acceleration is -10, we have

$$420=125t-\tfrac{1}{2}.10.t^2,$$

$$\therefore\ t^2-25t+84=0;$$

$$t=\frac{25\pm\sqrt{(625-336)}}{2}$$

$$=\frac{25\pm17}{2}=4 \text{ or } 21.$$

The reason for the double value of t is clear as before. The particle starts from O and arrives at A after 4 seconds, during which time its

velocity has been reduced to 125 − 40 or 85 cm. per second. It moves on with decreasing velocity until it is brought to rest at B after a further interval 85/10 or 8·5 seconds. Thus it takes 12·5 seconds to reach B from O. It now returns towards O under the acceleration 10 starting from rest at B and after a further interval of 8·5 seconds again passes through A. Thus it reaches A the second time 12·5 + 8·5 or 21 seconds from the start.

46. Falling Bodies. When a body is allowed to fall to the earth's surface from a point above it, it is found (1) that the acceleration is uniform[1], (2) that the acceleration is the same for all bodies.

The Experiments on which these statements are based will be described later. See §§ 65, 129.

This uniform acceleration of all bodies when falling from a given point is spoken of as the acceleration of gravity, or better, the acceleration due to gravity. It is usually denoted by the symbol g.

Again, Experiment shews us that the acceleration of a falling body differs slightly at different places on the earth; it is greatest at the poles and least at the equator. A body falls from a given height more rapidly at the pole than at the equator.

The value at the pole is 983·11 cm. per sec. per sec. and at the equator 978·10 cm. per sec. per sec.

At Greenwich the value is 981·17 cm. per sec. per sec.

Since 1 foot contains 30·48 cm. the value of g at Greenwich is 981·17/30·48 or 32·191 feet per sec. per sec.[2].

Hence we see that the formulæ of Section 45 are all applicable to the case of a falling body.

[1] This statement is only true for distances above the surface which are small compared with the radius of the Earth (4000 miles). It may be applied therefore without error to the experiments described in this book.

[2] See Example, p. 58. In working numerical examples we may use the values 980 cm. per sec. per sec. or 32 feet per sec. per sec.

When the body is projected downwards and starts with velocity u the acceleration g is in the direction of motion, and

$$v = u + gt \quad \text{.......................(i)},$$

$$s = ut + \tfrac{1}{2}gt^2 \quad \text{.......................(ii)},$$

$$v^2 = u^2 + 2gs \quad \text{....................(iii)}.$$

If the body be "dropped" it starts from rest so that $u = 0$.

If the body is projected vertically upwards and starts with the velocity u the acceleration g is opposite to the direction of motion and we have

$$v = u - gt \quad \text{.......................(i) } a,$$

$$s = ut - \tfrac{1}{2}gt^2 \quad \text{....................(ii) } a,$$

$$v^2 = u^2 - 2gs \quad \text{....................(iii) } a.$$

47. Problems on falling bodies.

(1) *To find the space passed over by a falling body in the* n*th second of its motion.*

Let s_1 be the space up to the beginning, s_2 up to the end of the nth second, then $s_2 - s_1$ is the space required.

Now, assuming the body to have been dropped,

$$s_1 = \tfrac{1}{2}g(n-1)^2,$$

$$s_2 = \tfrac{1}{2}gn^2.$$

$$\therefore\ s_2 - s_1 = \tfrac{1}{2}g\{n^2 - (n-1)^2\} = \tfrac{1}{2}g(2n-1).$$

(2) *A particle is projected upwards with velocity* u.

a. Find the height to which it will rise. Let this be H, the particle moves up till it reaches the height H, then it is instantaneously at rest and finally falls. Hence at a height H the velocity is zero,

$$\therefore\ 0 = u^2 - 2gH,$$

$$\therefore\ H = \frac{u^2}{2g}.$$

β. Find the time of rising. Let this be T_1, then at time T_1 the velocity is zero,

$$\therefore \quad 0 = u - gT_1;$$

$$\therefore \quad T_1 = \frac{u}{g}.$$

γ. *Find the time of falling.* Let this be T_2, then T_2 is the time of falling a height H.

$$\therefore \quad \tfrac{1}{2}gT_2^2 = H = \frac{u^2}{2g},$$

$$\therefore \quad T_2 = \frac{u}{g} = T_1.$$

Hence the times of rising and falling are the same. See also Example 4, p. 67.

δ. *Find the time at which it is at a height* h. Let this time be T_3.

Then $$h = uT_3 - \tfrac{1}{2}gT_3^2,$$

$$\therefore \quad gT_3^2 - 2uT_3 + 2h = 0.$$

$$T_3 = \frac{u \pm \sqrt{u^2 - 2gh}}{g}.$$

Hence T_3 has two values, one corresponding to the upward passage, the other to the downward passage of the particle.

ϵ. *Find the velocity at a height* h. Let this be v.

Then $$v^2 = u^2 - 2gh.$$

Hence if $h = 0$, or the particle is on the ground

$$v^2 = u^2, \quad v = \pm u.$$

Thus the velocity with which the particle reaches the ground is equal and opposite to that with which it starts.

48. Composition and Resolution of Accelerations.

A particle may have two or more accelerations communicated to it simultaneously. We proceed to determine the resultant effect.

PROPOSITION 19. *To find the resultant of two accelerations.*

Acceleration is measured by the change in velocity per second. If then OA, OB, Fig. 49, represent two accelerations communicated to a body, OA and OB represent the changes which take place in the velocity of the body per second.

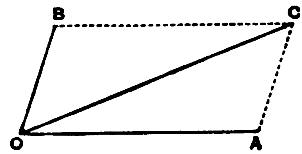

Fig. 49.

Complete the parallelogram $AOBC$. Then two velocities OA, OB have for their resultant OC. Thus the resultant change per second in the velocity of the body is OC. Hence OC is the resultant acceleration.

But OC is the diagonal of a parallelogram whose sides are OA and OB, the component accelerations. *Thus accelerations are combined according to the parallelogram law.* Hence the Propositions in Sections 30—31 about the Composition and Resolution of displacements apply to accelerations.

We may give *an alternative proof* of the above proposition thus.

Let PO, Fig. 50, represent the velocity of the body at any moment; let OA and OB represent the accelerations or changes per second which are to take place independently in the velocity. Draw AC equal and parallel to OB, then to find the velocity at the end of 1 second we have to combine with PO two velocities represented by OA and OB.

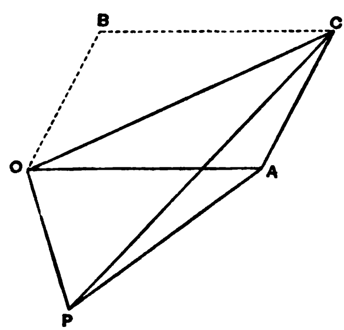

Fig. 50.

Combining PO and OA we get PA, combining with this OB we get PC. Thus PC represents the resultant velocity at the end of the second and PC is obtained by combining PO and OC. Thus OC is the change in velocity per second, that is, it is the resultant acceleration.

EXAMPLES.

MOTION WITH UNIFORM ACCELERATION.

1. Find the space traversed by a falling body in the eighth and tenth seconds of its motion.

2. With what velocity must a body be projected downwards in order to describe in 1 second a space equal to that described by a body falling freely in 2 seconds?

3. A bullet is shot up with a speed of 1000 feet per second, how high will it rise and after what time will it strike the earth again?

4. A body starts with a velocity 5 and has an acceleration of 2·5 in the direction of motion.

Find (i) its velocity after 3 seconds;
(ii) the space it has moved over in that time.

5. At the end of 3 seconds the acceleration of the body in the preceding question changes to 5 in a direction opposite to that of the motion; how far will it go before coming to rest?

6. A velocity of 5 is changed into one of 5 in a direction at right angles to itself, find the change in velocity and the acceleration supposing it uniform and that the change occurs in 10 seconds.
What will the velocity be at the end of 1 minute?

7. A stone is dropped over a cliff into water and the sound of the splash is heard after an interval of about $9\frac{1}{2}$ seconds; assuming the velocity of sound to be about 1150 feet per second, find the height of the cliff.

8. The velocity of a train passing two stations 1 mile apart are observed to be 30 and 50 miles an hour respectively; calculate its acceleration assuming it to be uniform.

9. A bullet shot up passes a point 1600 feet above its starting point on its upward and downward path respectively at an interval of half a minute, find its initial velocity and the height to which it rose.

10. A particle is moving under uniform acceleration with a velocity of 100 feet per second, at the end of 1 minute its velocity is 220 feet per second. How far will it move in 10 minutes and what will then be its velocity?

11. Draw the velocity-time curve for the bullet in question 9 and find hence or otherwise when it strikes the ground.

12. A body has a velocity of $3g$ and an acceleration g (*a*) in the direction of motion, (*b*) in the opposite direction. Find how far it moves in the first case before its velocity is doubled and in the second before it is halved. Find also the distances moved through in the two cases.

13. A body has an initial velocity 25 and an acceleration 1 opposite to the direction of motion. At what time will it have moved over half the distance it moves through before coming to rest and what will be its velocity then?

14. A body has an initial velocity u and an acceleration a. At what time after starting will it be moving with twice its initial velocity?

15. A stone is let drop from a given height and another is at the same instant projected vertically up to meet it. They pass at half the height, how high will the second stone rise and with what velocity does it start?

16. How long will a body, falling from rest, take to acquire a velocity of 96 feet per second?

17. With what velocity will a particle reach the ground if allowed to fall over a cliff 1156 feet in height?

18. A particle has an acceleration of 32 feet per second and passes a point at a distance of 1156 feet from the start with a velocity of 272 feet per second; find its initial velocity.

19. How high will a stone rise if projected up with a velocity of 250 feet per second?

20. A particle falls from a height of 78·48 metres, when will it reach the ground?

21. The speed of a train moving with uniform acceleration is doubled in a distance of 3 kilometres. It traverses the next $1\frac{1}{18}$ kilometres in 1 minute, find its initial speed and its acceleration.

22. A particle is thrown up with a velocity of 300 metres per second, how high will it rise and when will it strike the ground?

23. With what velocity will a particle reach the ground if it fall from a height of 400 metres?

24. The acceleration of a falling body is 981 cm. per sec. per sec.; find this in yards per min. per min.

25. How long will a falling body take to acquire a velocity of 100 metres per second?

26. The stick of a rocket reaches the ground 3 seconds after the explosion is seen, assuming the rocket to have been at rest when it burst, how high was it?

27. A train starts from rest and after 1 minute its speed is thirty miles per hour. Find the acceleration each second in feet per second supposing it uniform.

28. A moving point passes over 10 ft., 12 ft. and 16 ft. in three successive seconds, find its average velocity during the three seconds and determine whether or not it is moving with uniform acceleration.

29. The velocity of a body moving in a straight line is 32 feet per second at the end of 2 minutes and 40 feet per second at the end of 3 minutes. Find its initial velocity and its acceleration.

30. A train passes a station with a velocity of 30 miles per hour, and on passing the next, distant one mile, its velocity is 25 miles per hour. What is its acceleration?

31. A body moves in a straight line with uniform acceleration of 3·2 feet per second per second; find the time necessary to increase its velocity by a velocity of 15 miles per hour.

32. The position of a body moving with variable velocity is observed for each instant of its motion. Shew how to find from these observations the velocity and the acceleration of the body.

33. Prove that, when a body falls with a uniform acceleration, the difference between the square of the velocity at the beginning and end of the fall equals twice the product of the acceleration and the space traversed.

34. Two heavy bodies are dropped at the same time, one from a height of 50 feet, and the other from a height of 25 feet; find the height and velocity of the first, when the second touches the ground.

35. Prove that if a body is projected vertically upwards with the velocity of 64 feet per second, and 3 seconds afterwards a second body is let fall from the point of projection, the first body will overtake the second body one second and a half later at 36 feet below the point of projection, taking the acceleration of gravity to be 32 and neglecting the resistance of the air.

36. A balloon has been ascending vertically at a uniform rate for $4\frac{1}{2}$ seconds, and a stone let fall from it reaches the ground in $6\frac{1}{2}$ seconds after leaving the balloon: find the velocity of the balloon and the height from which the stone is let fall.

37. A particle moving with uniform acceleration in the direction of its motion has a velocity of 200 feet per second at the end of the third second, and of 260 feet per second at the end of the fourth second; what will be its velocity at the end of the fifth second? and what was the velocity at the instant from which the time has been reckoned?

38. A man on the bank of a river starts running with uniform speed just as the bow of a boat 60 feet long moving with uniform velocity is opposite to him. When the stern of this boat is opposite to him, the bow of a second boat, also moving with uniform velocity and of the same length as the first boat is just level with the first boat's stern, and just as the second boat passes the man, its bow is 20 feet behind that of the first boat. Find the distance apart of the boats when the man started.

39. Two trains of equal length, running at the rate of 40 miles an hour and 50 miles an hour respectively are approaching a level crossing. If the respective distances of the heads of the trains at the same instant from the crossing are 600 yards and 800 yards; find the greatest length that the trains can have so that there may not be a collision between them.

40. A train is moving at a rate of 60 miles an hour, and a gun is to be fired from a carriage window to hit an object which at the moment of firing is exactly opposite the window. If the velocity of the bullet be 440 feet per second, find the direction in which the gun must be pointed.

41. A man walks 2 miles in $\frac{1}{2}$ an hour, then drives 5 miles in $\frac{3}{4}$ hour, afterwards he travels 10 miles in an hour, and completes a further 8 miles of his journey in 20 minutes. Express his second speed in miles per hour, and his last speed in feet per second; and find what his average speed on the whole journey has been.

42. A man is running with a velocity of 6 miles per hour in a shower of rain which is descending vertically with a velocity of 11 feet per second. Find the tangent of the angle which the apparent direction of the rain makes with the horizon.

43. A certain mark on the circumference of a flywheel 6 ft. in diameter passes a fixed point three times every minute. Find the velocity of the mark and the angular velocity of the wheel.

44. A point moving in a straight line describes a space x in a time t, and its velocities at the beginning and end of the time are u and v. Find an expression for the mean acceleration of the point, and if its acceleration is constant, prove that $2x=(u+v)\,t$.

45. ABC is a triangle right-angled at C. Points start from A and B at the same instant and move towards C with uniform acceleration inversely proportional to AC and BC respectively. Shew that their least distance apart will be $\dfrac{AC^2 \sim BC^2}{AB}$,

46. A stone is thrown vertically upwards and returns to the point of projection after 5 seconds; find the greatest height to which it rises and its velocity, on its return, at the point of projection.

47. A stone is thrown vertically upwards and just reaches a height 80 feet above the point of projection. Find its velocity, on its return, at the point of projection and the whole time taken.

48. A uniformly accelerated body passes two points 6 feet apart in $\frac{1}{4}$ second; 4 seconds after reaching the first of these points the body had a velocity of 110 feet per second; find the velocity and acceleration of the body.

49. A falling body passes two points 10 feet apart in $\frac{1}{2}$ second: it subsequently passes two other points also 10 feet apart in $\frac{1}{10}$ second. If the acceleration due to gravity is 32 feet per second per second, find the distance between the first and the last of these four points.

50. If the measure of an acceleration is 528 when a yard and 6 seconds are the units of length and time, find its measure when a mile and an hour are the units of length and time.

CHAPTER IV.

MOMENTUM.

49. Mass. In the two following chapters we shall endeavour to obtain from experiment some notion of the meaning of the terms Mass and Force as used in Mechanics and to arrive at certain laws which express the results of the experiments. These laws, known as Newton's Laws of Motion, are from this point of view, generalizations from simple observations. Having arrived at these generalizations, we can start afresh and, assuming the laws as true in all cases, can deduce from them the motion of bodies under various complex circumstances. This is done in Chapter VI. and the following chapters, in which Mechanics is treated as a Deductive Science based on Newton's Laws as Axioms.

We are now about to consider certain effects produced by a moving body which we may treat as a particle. These effects we shall find may be different for different bodies; from the consideration of them we may obtain a definite idea of the meaning of the term **Mass** in Mechanics.

We can recognize in bodies in many ways a property which depends partly on their size and partly on the substance of which they are composed. Thus, if we take two balls of iron of considerably different sizes and hang them up by long strings of the same length, a very slight effort is sufficient to give a considerable velocity to the small ball, while a strong push is needed to displace the large ball appreciably; the two balls are said to differ in mass. Or again consider two casks of the same size, the one filled with sand, the other with feathers; a slight kick is sufficient to start the second cask rolling, a vigorous shove will hardly stir the former; we say that the mass of the sand is greater than that of the feathers.

A heavy flywheel properly mounted on ball bearings continues to rotate for a long time when set in motion, an

appreciable effort is needed however either to stop it, or to start it when at rest; the flywheel is said to have mass, and the greater the mass the greater the effort needed to stop it in a given time. Two identical lumps of metal when suspended by a fine string over a light pulley remain at rest; a downward push applied to one will start them moving, and when started they continue to move, but a stronger effort is needed if the bodies suspended be large than is required if they be small.

These and similar observations lead us to recognize that property of bodies to which the name of **Mass** has been given. We are able as we shall see to compare the masses of two bodies and shall find that for a given homogeneous substance the mass of a body depends on its volume, while for bodies of given volume the mass depends on the substance of which the bodies consist and on its physical state.

50. Experiments on the Measurement of Mass.

Newton describes in his *Principia* certain experiments on the collision or impact of bodies from which he draws several important conclusions. The observations just described lead us to the conception of mass as a fundamental notion; experiments based on those of Newton enable us to give definiteness to the idea.

Newton in his experiments employed two spherical balls suspended from two points in the same horizontal line by parallel strings of such lengths that when at rest the balls were in contact and their centres were at the same distance below their points of suspension (Fig. 51). The experiments consisted in drawing the balls apart to various small distances, and then allowing them to fall simultaneously; the balls then struck each other at the lowest point of their swing, and the velocities with which they impinged were calculated[1]; the positions to which the balls rose after impact were observed and from this observation their velocities after impact were obtained; from the relation between these velocities various important deductions can be drawn.

Fig. 51, taken from the *Principia*, shews the arrangement adopted by Newton. A body such as the ball A or B, suspended so as to be able to swing backwards and forwards

[1] See Section 146.

about its position of rest is called a simple pendulum, and in the deductions from the experiments we make use of two laws discovered by Galileo, as to the motion of a simple pendulum[1].

According to the first law, if the ball of a simple pendulum be pulled aside a moderate distance, so that the string is

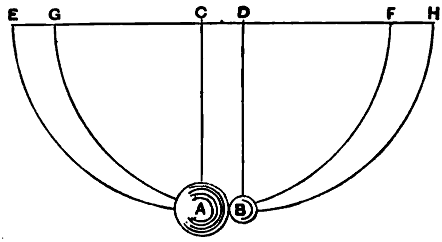

Fig. 51.

inclined to the vertical at a small angle, when the ball is released it will take very approximately the same time to reach its equilibrium position, in which the string is again vertical, whatever be its starting point, provided only that the angle the string makes originally with the vertical be not large. Thus if the two balls A and B be drawn a short distance apart and let fall simultaneously, since the distances between the points of suspension and the centre of each ball are the same, they will always impinge at the lowest point.

If the ball A be drawn a very short distance aside and released, its velocity as it passes through the equilibrium position will be small; if the original displacement be larger, the ball after release will arrive at its lowest point in the same interval of time as before, but since in the second case it has in that interval traversed a greater distance than in the first case the velocity with which it reaches it will be greater.

The second law referred to above enables us to calculate what the velocity is.

Let P, Fig. 52, be the point from which the ball A is allowed to start. Join PA. Then it can be shewn that the velocity with which the ball will reach its equilibrium position is

[1] Experiments to verify the laws are described later (see § 130).

proportional to the distance PA. If the ball A, after striking the second ball, rise to Q its velocity is proportional to AQ, we

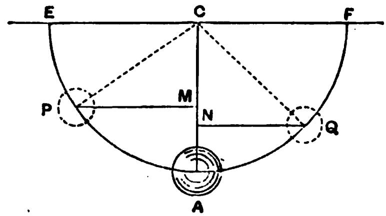

Fig. 52.

can compare the velocities before and after impact by measuring the lines PA and QA.

Let PM and QN be horizontal lines drawn from P and Q respectively to meet the vertical line CA in M and N. Then if the arcs PA and QA be small, the ratio of PA to QA is nearly the same as that of PM to QN. The velocities before and after impact are approximately proportional to PM and QN.

We could perform our experiments with the apparatus used by Newton, an arrangement however which has been devised by Professor Hicks of Sheffield will serve better. It is called by him a Ballistic Balance.

51. Hicks' Ballistic Balance. The apparatus is shewn in Fig. 53. It consists of a rectangular wooden framework $ABCD$, about 100 cm. long and 125 cm. high. The bar AB is horizontal, AC and BD are vertical, and the framework is arranged so as to stand securely on a table. Four parallel bars EF about 20 cm. long can be adjusted across AB.

From these bars two carriers G, H, are supported by fine wires or threads as shewn in the figure, the carriers are small rectangular pieces of wood, the lengths of the wires and the positions of the bars EF are adjusted so that the planes of the carriers as they swing are always horizontal, and the two carriers strike each other perpendicularly when each is at the lowest point of its swing. The ends of the carriers which come into contact are fitted with some sharp-pointed pins so that the carriers after impact adhere together.

LMN is a horizontal bar fitted below the carriers and carrying two scales, one for each carrier. A vertical pointer is fixed to each carrier and moves over the corresponding scale, which is adjusted so that the pointer reads zero when the carriers are in contact. If a carrier is pulled aside and let

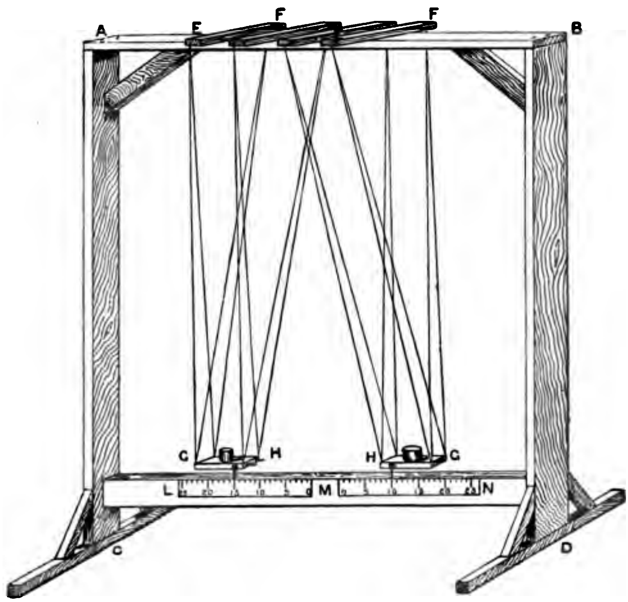

Fig. 53.

go, the velocity with which it reaches its lowest point is proportional to the horizontal distance through which it has been displaced, and this is given directly by the scale reading of the pointer at the starting point. The actual value of the velocity will depend on the dimensions of the instrument; if the vertical distance between the carrier and the points of support of the wires be 109 centimetres, a displacement of 1 centimetre along the scale can be shewn to give rise to a velocity of 3 cm. per second; each centimetre of the scale corresponds therefore to a velocity of 3 centimetres per second.

By attaching a string to each carrier, it can be pulled aside any required distance, on releasing the strings simulta neously the two carriers are set in motion and impinge at the lowest point of their swings. By passing the strings over pulleys they can both be fastened to the same clip so as to secure that the two carriers start simultaneously.

52. Experiments with the ballistic balance.

EXPERIMENT 9. *To determine the condition that two bodies may have equal masses.*

(*a*) It has been stated previously that the masses of two equal volumes of any homogeneous material are equal. Take two equal volumes of lead, say two cubes some 3 or 4 centimetres in edge. Place one on each carrier and displace the two carriers equally, say 2 cm.; release them simultaneously; they meet at the bottom and it will be found that both are reduced to rest. If two equal volumes of iron, silver, etc. be used the result will be the same. Repeat the experiment but let the displacement be greater, still keeping it the same for both; the result is always the same, the masses are reduced to rest. Thus two identical lumps of matter when they impinge on each other directly with the same velocity are reduced to rest.

Now take a lump of iron and a lump of lead of the same volume, place one on each carrier and repeat the experiment; the impact will no longer result in rest, the lead lump will continue to move forwards though with reduced velocity, the iron will be driven back. The iron is said to have less mass than an equal volume of lead.

(*b*) Take a smaller piece of lead and repeat the experiment, the velocity after impact will be less than before—with a still smaller piece of lead the velocity may even be reversed. By adjusting the volume of the lead we can again obtain the result that there is no motion after impact. When this has been secured the two masses of iron and lead are equal.

DEFINITION OF EQUAL MASSES. *The* **Masses** *of two bodies are* **equal,** *if, when the bodies impinge on each other directly with equal velocities, they are reduced to rest.*

53. The comparison of Masses. If the volume of the lead used in EXPERIMENT 9 (*b*) be measured it will be found to be about 78/114 of that of the iron.

Thus, according to this definition of equal masses, a given volume of iron has the same mass as 78/114 of that volume of lead. Hence the masses of equal volumes of iron and lead are as 78 to 114. We have thus a means of comparing the masses of two bodies.

In this experiment the bodies impinge with equal velocity, they are reduced to rest, and we say that their masses are equal.

Consider now what happens if the velocities with which the bodies meet be not equal.

Place on the carriers two equal lumps of lead. Draw back one carrier further than the other and release them. On impact the carrier which has the greater velocity continues to move onwards, the motion of the other is reversed. Replace one of the lumps of lead by an iron lump of equal volume and draw it back further than the lead lump so that on impact the iron lump may have the greater velocity; the velocity after impact will be less than it was when the two met with the same velocity, and by careful adjustment positions can be found for the carriers such that they remain at rest after impact.

Thus for example if the lead lump be displaced 7·8 cm.[1], and the iron 11·4 cm., then after impact the carriers will be at rest, the same will be true if the displacements respectively be 3·9 and 5·7 cm. or 15·6 and 22·8 cm.

Now these numbers are to each other respectively in the ratio of 78 to 114, that is, in the ratio of the mass of the iron to the mass of the lead.

Again, the velocities with which the two impinge are proportional to the displacements, and we find in this case that there will be rest after impact provided that we satisfy the relation given by the equation,

$$\frac{\text{Final Velocity of Lead lump}}{\text{Final Velocity of Iron lump}} = \frac{\text{Mass of Iron}}{\text{Mass of Lead}},$$

[1] In these numbers no allowance has been made for the mass of the carriers. Should this be appreciable compared with the masses of the bodies placed on them some correction will be required.

or writing u_1, u_2 for the two velocities, m_1, m_2 for the two masses, provided that

$$\frac{u_1}{u_2} = \frac{m_2}{m_1},$$

or $$m_1 u_1 = m_2 u_2.$$

Whenever then this relation is satisfied the carriers after impact will come to rest.

Thus we can use the ballistic balance to compare two masses by determining the velocities with which they must impinge directly in order to be reduced to rest, for we have the result that, *When two bodies are caused to impinge directly so as to adhere together and are reduced to rest by the impact their masses are inversely proportional to the velocities with which they impinge.*

Moreover we can shew, by direct experiment, that the results are not modified by altering the shape of the lumps of matter used. If we determine the ratio of the masses of a lump of iron and a lump of lead we may alter the shape of the lead by hammering it, or in any other way. If we do not remove any of the lead its mass as determined by the ballistic balance in terms of that of the iron lump remains unchanged.

Other and simpler methods of comparing masses will be given later. (*Statics*, § 59.)

54. The Unit of Mass. We have explained § 11 that we assume as the unit of mass the mass of a certain lump of platinum called a kilogramme. We may imagine then that we use this in one of the carriers of the ballistic balance, and give it a certain velocity; we can determine in terms of this the mass of any other body by finding the velocity which must be given to that body in order that it may be reduced to rest by impact with the standard mass.

Moreover the ratio of the masses of two bodies is found to remain the same from day to day; and is not altered by carrying the bodies from point to point of the earth's surface. The mass of any given body has at all times and places a constant ratio to that of the standard; if we assume its mass to be a definite property of the standard then the mass of every other body is a definite property of that body.

55. Mass and Quantity of Matter. Suppose we determine by the ballistic balance the mass of a body; remove part of the body and again measure its mass; it will be found that the mass is reduced.

Now in ordinary language we should say that we had taken away some of the matter of which the body is composed. We thus see how it comes about that mass is looked upon as measuring the quantity of matter in a body, and what is meant by the statement, that the mass of a body is the quantity of matter of which it is composed.

It may however be convenient sometimes to employ the term "quantity of matter" as identical with the term "Mass," to say that in Dynamics the quantity of matter in a body is measured by its mass; the mass can be compared with the unit mass by means of a ballistic balance or in some equivalent manner.

56. Momentum. The experiments with the ballistic balance have led us to recognize a quantity in mechanics which depends on the product of the mass of a moving body and its velocity. The motion which a given body can communicate by impact to another body depends on this quantity. This quantity is called **Momentum** and we shall find that it is of fundamental importance.

DEFINITION. *The Momentum of a body is the product of its mass and its velocity.*

Thus if a mass of m grammes be moving with a velocity of v centimetres per second its momentum is mv.

The unit of momentum therefore is the momentum of a mass of 1 gramme which moves with a velocity of 1 cm. per second.

Various names have been suggested for the unit of momentum but none of them has received general acceptance. When then, we say that the momentum of a body is 10 we mean that it has 10 units of momentum; its momentum therefore is the same as that of a mass of 10 grammes moving with a velocity of 1 cm. per second or of 1 gramme moving with a velocity of 10 cm. per second, or of x grammes moving with a velocity $10/x$ cm. per second.

In dealing with momentum we must remember to take into account the direction in which the body is moving. Thus a billiard ball which impinges directly on the cushion has the direction of its motion reversed by the impact; if we agree to call its velocity and its momentum positive before the impact, we must call them negative afterwards.

57. Condition of rest after Impact. We can now express in terms of momentum the condition that two bodies which meet directly and adhere should be brought to rest by the impact. From Section 53 we know that the condition is that

$$m_1u_1 = m_2u_2.$$

Now m_1u_1 is the momentum of the first body, m_2u_2 that of the second; the condition then for rest is that the momenta of the two bodies should be equal in magnitude and opposite in direction. If we call the momentum of one body positive that of the other will be negative, we may say then that the condition for rest after impact is that the total momentum of the system before impact should be zero.

The momentum after impact is zero, so that we see that in this case there is no change in the momentum of the system.

***58. Further experiments with the ballistic balance.** By the previous experiments we have determined the condition that the bodies should be reduced to rest by the impact. We now wish to determine the velocity with which they will move if this condition be not fulfilled, we still suppose that the two carriers adhere together after the impact.

EXPERIMENT 10. *Two masses impinge directly and adhere; to determine by experiment the relation between their velocities before and after impact.*

Take two equal masses, such as the two lumps of lead used in EXPERIMENT 9 and place one in each carrier of the balance. Let one mass remain at rest in its lowest position. Displace the second carrier a measured distance a, say 20 centimetres, and let it go. After the impact the two carriers move to-

gether. Observe the extreme distance to which they swing; let it be b centimetres. Then it will always be found that

$$b = \tfrac{1}{2}a,$$

thus if the original displacement of the first mass be 20 cm. the joint displacement after impact will be 10 cm.

Now the first displacement measures the velocity with which the one mass strikes the other, the second measures the velocity with which the two masses move after impact. We thus see that in this case the velocity is halved by the impact, but at the same time the moving mass is doubled, the momentum therefore of the system remains the same, the one mass loses momentum while the other gains an equal amount, there is no change in the total amount, it is distributed between the two instead of being entirely in the one.

Now, however the masses on the carriers be changed, it will be found that this law always holds; the total momentum is unaltered in all cases.

If m_1, m_2 be the masses, u_1 the velocity with which m_1 strikes m_2 at rest, and u the common velocity after impact, then we shall find that

$$(m_1 + m_2)\, u = m_1 u_1,$$

the momentum of the two after impact is equal to the sum of the momenta before; the velocities u_1 and u are measured respectively by the original displacement of the first mass and the joint displacement b of the two after impact. Make a series of observations of b giving a different values such as 2, 4, 6, 8, 10, etc. cm. It will be found in all cases that the ratio of b to a is a constant and that this ratio is equal to the ratio of m_1 to the sum $m_1 + m_2$.

Make another series of observations in which both the masses m_1 and m_2 are in motion before impact. Displace m_1 a distance a_1 and m_2 a less distance a_2 in the same direction; on releasing the two simultaneously m_1 will acquire a greater velocity than m_2 and will overtake it at the bottom of the swing; after impact the two bodies will move on together with the same velocity u and it will be found that the momentum of the two is the sum of the original momenta so that

$$(m_1 + m_2)\, u = m_1 u_1 + m_2 u_2,$$

or, if b be the first displacement after impact

$$(m_1 + m_2)\, b = m_1 a_1 + m_2 a_2,$$

$$b = \frac{m_1 a_1 + m_2 a_2}{m_1 + m_2}.$$

Make a series of observations for different values of a_1 and a_2 and verify this formula.

A similar result can be obtained when the two bodies move in opposite directions before they impinge, only in this case we must treat the momenta of the two as opposite in sign. The combined system will after impact move in the direction of motion of that body which before impact had the greater momentum; we shall have the equation

$$(m_1 + m_2)\, u = m_1 u_1 - m_2 u_2,$$

satisfied

or

$$b = \frac{m_1 a_1 - m_2 a_2}{m_1 + m_2}.$$

In all the above cases we see that there is no change of momentum produced on the whole.

***59. Impact of elastic bodies.** This same result is true and can be verified by the ballistic balance even though the carriers rebound from each other after impact; the observations[1] are rather more troublesome because there are two quantities b_1 and b_2 the displacements of the two carriers to observe.

We shall find that it is impossible without further knowledge as to the properties of the material to calculate *a priori* the values of b_1 and b_2 from a knowledge of the original displacements and the masses, but it will follow the results of observation that the values of b_1 and b_2 are always such that the momentum of the system remains unchanged.

They satisfy the relation

$$m_1 b_1 + m_2 b_2 = m_1 a_1 + m_2 a_2,$$

[1] In such experiments the pins or clips arranged to fasten the carriers together after impact are removed; the carriers therefore are free to rebound and the nature of the rebound will depend on the material of which they are composed as well as on their original momenta.

so that, if we call v_1, v_2 the velocities after impact which correspond therefore to b_1 and b_2 and u_1, u_2 the velocities before impact corresponding to a_1 and a_2, we have

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2.$$

Thus when two bodies impinge directly the momentum remains unchanged.

Newton's experiments already referred to were designed in part to verify this law, in part also to determine another relation between v_1 and v_2 from which, when combined with the above, it might be possible to calculate v_1 and v_2. We shall return to this point again, for the present we are concerned with the law of the permanence or conservation of momentum.

60. Change of Momentum—Impulse. In each of the above experiments, while the momentum of the whole system has remained unchanged the momentum of each body has been altered, there has been a transference of momentum from one body to the other, the one has gained what the other has lost.

Thus, if in EXPERIMENT 10 we take two equal masses m, the velocity of the striking mass is changed by impact from u to $\frac{1}{2}u$; its momentum was mu and it becomes $\frac{1}{2}mu$; its loss of momentum therefore is $\frac{1}{2}mu$. The velocity of the second mass is changed from 0 to $\frac{1}{2}u$; its momentum therefore alters from zero to $\frac{1}{2}mu$, its gain is $\frac{1}{2}mu$ which is equal to the loss of the first mass.

Now this law is quite general, for with the notation of Section 58, the loss of momentum of m_1 is

$$m_1u_1 - m_1v_1,$$

or

$$m_1(u_1 - v_1).$$

The gain of momentum of m_2 is

$$m_2v_2 - m_2u_2,$$

or

$$m_2(v_2 - u_2).$$

Now we have

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2.$$

Hence

$$m_2 v_2 - m_2 u_2 = m_1 u_1 - m_1 v_1,$$

or

gain of momentum of m_2 = loss of momentum of m_1.

Thus momentum is transferred unchanged in amount from the one mass to the other.

The name **Impulse** has been given to the whole change of momentum of a body.

DEFINITION. *The gain of momentum of a body is called* **Impulse.**

Thus in the experiments in Section 56 the second mass *gains* an amount of momentum $m_2(v_2 - u_2)$, its Impulse I_2 therefore is given by

$$I_2 = m_2(v_2 - u_2).$$

The first mass *loses* an amount of momentum $m_1(u_1 - v_1)$, its impulse I_1 is given by

$$I_1 = -m_1(u_1 - v_1)$$
$$= m_1(v_1 - u_1),$$

and in this case I_1 is negative.

Moreover we have the result that

$$I_2 = -I_1$$

or

$$I_1 + I_2 = 0,$$

that is, the total impulse is nothing.

61. Transference of Momentum. Thus we have learnt from these experiments to consider momentum as a property of a moving body which we can measure in a definite manner. This property can be communicated through impact by one body to another; when such transference takes place under circumstances similar to those of the above experiments, there is neither loss nor gain of momentum, the one body gains what the other loses, the amount of momentum transferred is unaltered. Now we find that this law is of general application; momentum can be transferred from one body to another

in other ways than by direct impact; whenever such action goes on in an unimpeded manner there is no loss in the amount of momentum transferred, the gain of the one body is equal to the loss of the other.

The words "in an unimpeded manner" above are of importance. Consider two isolated particles and suppose that there are no other bodies near which can in any way affect their mutual action, this action is unimpeded; the two particles will move towards each other in such a way that in any given time they each gain equal amounts of momentum in opposite directions. The total change of momentum will be zero, the impulse of the one particle is equal and opposite that of the other.

But we cannot in practice secure that the above conditions shall be satisfied; it is impossible to obtain two particles free from the influence of all other bodies. Our experiments must be performed in the presence of the earth, and this may have an effect on the change of momentum produced. In the cases of impact with which we have been dealing no such effect is produced as we shall see later, the action is unimpeded.

Take another example. If a stone be held at a distance from the earth it has no momentum relatively to the earth; on releasing it it falls and acquires momentum, there is here a gain as far as we can observe of momentum; in reality we believe that there is no gain, for the earth has acquired momentum in the opposite direction equal to that of the stone.

We cannot of course verify this by direct experiment, we have no means of determining whether the earth moves towards the stone or not; the velocity acquired by the earth would be excessively small, for its mass is enormous compared with that of the stone.

Thus if a stone fell from a height of 5 metres it would on reaching the ground have a velocity of about 1000 centimetres per second ($v^2 = 2gs = 2 \times 981 \times 500$, $v = 1000$ approximately). Suppose the mass of the stone to be 1 kilogramme or 10^3 grammes; the mass of the earth is about 5×10^{27} grammes. Thus the momentum of the stone is $10^3 \times 10^3$ or 10^6 units of momentum. Hence the velocity of the earth is $10^6/5 \times 10^{27}$ or 2×10^{-22} cm. per second. Now there are rather more than 3×10^7 seconds in a year; hence if the earth were to continue to move for a year with the velocity thus acquired it would only traverse 6×10^{-15} or ·000,000,000,000,006 of a centimetre.

62. Conservation of Momentum. We may sum up then the results of the experiments and our discussion of them with the statement that by the mutual action between two bodies momentum can be transferred from the one to the other. When this mutual action is unimpeded the momentum transferred remains unchanged in amount.

This principle is known as the Conservation of Momentum.

CHAPTER V.

RATE OF CHANGE OF MOMENTUM. FORCE.

63. Force. The experiments described in the last chapter have shewn us how to attach a definite meaning to the term mass as used in Mechanics and have led us to recognize Momentum as a fundamental property of a moving body. We have seen also that momentum is transferred without loss from one body to another by impact and have given a name, Impulse, to the change in momentum.

Now the velocity of a body is uniform and its momentum constant when it moves in a straight line and passes over equal spaces in equal times. If we observe the velocity of any body[1] by noting its positions at given intervals of time, we find that there are very few cases in nature in which the velocity is uniform; in nearly all the velocity is variable, the body has acceleration.

Thus in most cases the momentum of a moving body changes. We are now about to investigate in certain cases the rate at which this change takes place. This rate of change of momentum has received a name, it is called **Force**.

DEFINITION. *The Rate at which the Momentum of a moving body changes is called* **Impressed Force.**

In the experiments just described the change of momentum has been a sudden one. The Impulse has occurred in a very

[1] As has been already explained, § 15, we are at present dealing only with bodies which for the purposes of our investigation may be treated as particles.

brief interval of time; suppose now we consider a series of very small impulses applied for brief consecutive intervals of time; the small changes of momentum occurring during each impulse will add up; at the end of a finite time a finite change of momentum is produced, but the change has been a gradual one, the motion has taken place under an impressed force.

Force is often looked upon as something external to the body acting on it and causing it to move. Now when we say that a body is moving under the action of a force all that we can observe is a change in the momentum of the body. We are however often not content with the simple observation that the motion of the body is changing in a definite way, we endeavour to assign a cause for this change of motion and call this cause Force. We look upon the change in momentum as due to the mutual action between the moving body and some other body, the Earth for example, which can influence it, and we say that the change of momentum is due to this action. Our sensations give us some knowledge of a mutual action between ourselves and other bodies which if not impeded is followed by motion, and it was doubtless to this muscular sense that the idea of force was originally attached. But our sensations alone cannot enable us to measure Force. We cannot prove that Force as measured by the rate of change of momentum corresponds to our muscular sensations. For our purposes we therefore dismiss at once the notion of there being any connexion between the two. Force as a *Cause of Motion* we have not here to consider; it will suffice for us to define it as *Rate of change of Momentum* and proceed to examine certain simple cases of motion with a view of seeing what deductions we can make from them as to the relation between Force and Motion.

In this we are adopting a historical method of procedure. The experiments we are about to describe resemble those by which Galileo established some of the fundamental laws of Mechanics; their discussion will lead us naturally to the consideration of Newton's Laws of Motion given by him in the first pages of the Principia as the fundamental Axioms of the subject.

We are about to deal with the rate of change of momentum of bodies and to consider in the first case that of falling bodies; we shall shew how to express the rate of change of momentum of a body in terms of its mass and its acceleration.

64. Measurement of Force.

PROPOSITION 20. *The Rate of change of momentum of a body is the product of its mass and its acceleration.*

Let the velocity of the body initially be u cm. per second, and suppose that after t seconds it is v cm. per second, and that the acceleration is uniform and equal to a cm. per second per second.

Then the momentum originally is mu, after t secs. it is mv. Thus

The change of momentum in t seconds is $mv - mu$ and the change in 1 second is $(mv - mu)/t$ or $\dfrac{m(v-u)}{t}$.

But $$v = u + at.$$

Hence $$\frac{v-u}{t} = a,$$

and the change in momentum per second is ma.

But when a quantity changes uniformly the change per second measures the rate of change.

Thus if F be the impressed force we have $F = ma$.

Hence the rate of change of momentum is ma.

When the acceleration is variable the same expression holds, for we deal with variable acceleration by supposing it uniform for a very short space of time and considering what takes place in the limit when the time is indefinitely diminished; now the above formulæ are true when t is indefinitely small and a variable.

65. The acceleration of a falling body. When a body falls it moves with a continually increasing velocity: the motion is accelerated. Observation shews as we have already stated that the acceleration is a uniform one. This acceleration is spoken of as the acceleration due to gravity; its value in England is about 981 cm. or 32·2 feet per second per second. The most direct way of proving this would be by observing the distance a body falls through in various intervals of time, this method is not easy to put into practice, for unless the intervals of time are very short the space traversed and the velocities acquired become considerable and difficult of measurement, while if the times are short it is difficult to measure them exactly.

The difficulty was avoided by Galileo, who observed the time taken by a ball to roll down an inclined plane, and from that inferred what would happen if it fell freely; it is met in another way, as we shall see shortly, in Atwood's Machine. We will however first give some experiments on bodies falling freely.

In experiments on Motion we need some arrangement for measuring short intervals of time. For many purposes a good stop-watch will serve; this is fitted with a long seconds-hand, the dial is divided into seconds and these subdivided into fifths and observations can thus be made to the fifth of a second.

In other cases a pendulum which ticks once a second is useful. It is not necessary that there should be any clockwork attached. A heavy pendulum once started will continue to move sufficiently long for an experiment without the aid of a spring.

It is convenient for many purposes to have an arrangement attached to the pendulum, by which a circuit carrying an electric current may be made or broken—as is most convenient—once a second. The current may be made to ring a gong which thus sounds at intervals of a second

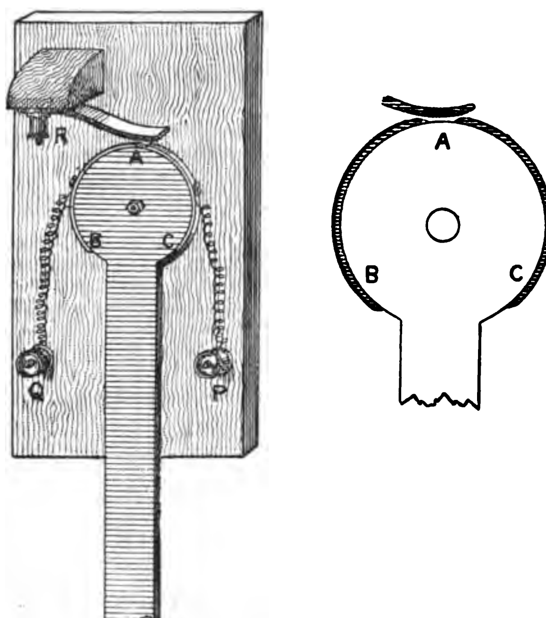

Fig. 54.

and marks the time more definitely than the ticks. Fig. 54 shews a device which is useful.

The end of the pendulum is an arc of a circle, on the edge of this, at A the top of the arc are two strips of thin brass insulated from each other. These are connected by flexible wires to two binding screws P and Q. A strip of thin brass connected to a binding screw R makes contact with the top of the pendulum rod at a point just above its point of support. A strip of paper or thin insulating material (shewn in black in the section at the side of Fig. 54) is pasted over the strip of brass AB leaving only a narrow portion of the brass near A exposed. The screw R is connected to the battery, Q to the electro-magnet and P to the bell. The second poles of the electro-magnet and of the bell are connected together and to the second pole of the battery.

When the pendulum is pulled to the right contact is made between R and P by means of the spring and the strip AC; the current passes through the electro-magnet which is magnetised and can support an iron ball or a wooden ball which has a small piece of wire attached. When the pendulum is released this contact is maintained until the bob reaches the lowest point of its swing, and at the moment contact is broken between the spring and AB it is made between the spring and AC; the bell is rung and as it rings the ball drops; the contact with AC, since the brass exposed is very narrow, is only maintained for a very brief time, the gong sounds once or perhaps twice and then is silent until the pendulum again passes through its lowest position; the time of swing of the pendulum can be adjusted by altering the position of the bob and we thus have a means of marking seconds, half-seconds or other intervals after the ball is dropped.

A tuning-fork or a vibratory bar may also be employed to measure small intervals of time. The prongs of the fork when it is struck vibrate and each vibration occupies the same time; the period of vibration can be measured. Suppose now a light metal style is attached to one prong and a piece of smoked glass is held so that the point of the style is just in contact with it, if the glass be raised or lowered vertically, the style will trace a straight line along it.

Let us now suppose the fork is set in motion so that the vibrations take place in a horizontal plane and the glass moved uniformly past it, the straight line becomes a regular undulatory curve. Each loop of the curve is of the same size and it cuts the straight line drawn by the fork when at rest in points which are at equal distances apart. If the fork makes 20 complete vibrations per second each of these distances will correspond to the one-fortieth of a second and will measure the distance traversed by the plate in the fortieth of a second. If the motion of the plate be not uniform the distances will not be equal and the loops of the curve will not be alike, each space will however be the distance traversed by the plate in the corresponding fortieth of a second. By measuring the spaces we can deduce various consequences as to the motion.

EXPERIMENT 11. *To shew that a body falling freely passes over approximately* 490·5 *centimetres*—16·1 *feet*—*in the first second of its motion from rest.*

The seconds pendulum just described is used for this, the bob is adjusted so that the pendulum makes one complete oscillation in two seconds, it thus passes through its equilibrium position once a second and at each transit the gong is sounded once. The binding screws P and R are connected with an electromagnet and battery and the pendulum is drawn aside and held by a string so that connexion is made between P and R through the brass strip AB and the spring.

The electro-magnet which is thus magnetized and supports an iron ball is attached to a light wooden frame which can be raised by means of a string to any desired height; a measuring tape is also attached to the frame and the height to which it is raised can be easily measured.

Start the pendulum; then, as the pendulum reaches its lowest position the electric circuit round the magnet is broken, thus releasing the ball, and the gong is sounded at the same time. Raise the electro-magnet and adjust its height until the ball on falling strikes the floor simultaneously with the second stroke of the gong; this coincidence can be estimated with considerable accuracy. Measure the height of the electro-magnet; the interval between the two sounds is one second and during this interval the ball has fallen through the height just measured.

It will be found on making the measurement that the height is about 490·5 centimetres or 16·1 feet.

If we attempt to use this method to find the distance fallen through in a longer time, say 2 seconds, we find that the distance is too great for measurement in the Laboratory—it would be $4 \times 490{\cdot}5$ or 1962 cm. In order then to find how the space traversed by a body falling freely varies with the time we must have recourse to some method of measuring small intervals of time such as that described above.

EXPERIMENT 12. *To shew that the space passed over by a body falling freely from rest is proportional to the square of the time of motion.*

A massive tuning-fork, Fig. 55, making say 20 vibrations per second is mounted so that its prongs vibrate in a horizontal

plane and a light style is attached to one prong. The style may conveniently be a bristle from a brush and its end should point downwards. A glass plate of considerable mass

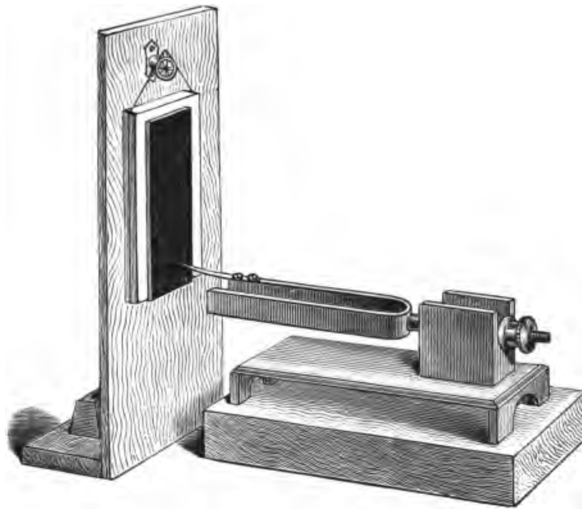

Fig. 55.

is supported, with its lower edge just below the point of the style, by a single string passing over a pulley and the plate can be allowed to drop by burning the string above the pulley. The plate is supported in such a way that the upper part of its front surface is slightly tilted forwards; hence as the plate falls it comes almost immediately into contact with the style. The back of the plate rests against two narrow vertical strips of wood; it is thus prevented from swinging in its fall. The front of the plate is coated with lamp black and the style as the fork vibrates marks a sinuous trace on the falling plate.

The style, if the fork had been at rest, would have traced a vertical line on the plate. This line can be drawn in

afterwards and a trace such as that shewn in Fig. 56 is obtained; the point A which was opposite to the style before the plate was started is also marked.

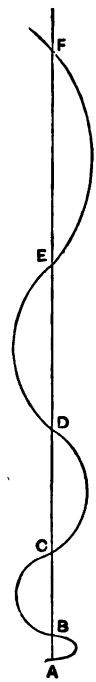

Fig. 56.

In such a case the spaces AB, AC, AD etc. represent the distances traversed in 1, 2, 3 etc. twentieths of a second and these distances can be measured; it will be found that approximately

$$AC = 4\ AB = 2^2 AB$$
$$AD = 9\ AB = 3^2 AB$$
$$AE = 16 AB = 4^2 AB,$$

and these are the spaces traversed in 2, 3, 4, etc. twentieths of a second. Thus the spaces traversed are proportional to the squares of the time.

We have already seen that when a particle moves with uniform acceleration it traverses spaces which are proportional to the squares of the times. Hence we infer that a falling body moves with uniform acceleration.

In making the experiment it will not usually happen that the trace passes accurately through the point A. This will affect the measurements of the distance traversed, but except in the case of the first few intervals the error will not be large. We can eliminate it from the result and use the experiment to shew that the acceleration is uniform by a method due to Prof. Worthington. For let l_1 and l_2 be the distances traversed in any two equal consecutive intervals of time t, then if the acceleration is uniform l_1/t and l_2/t will be the velocities at the middles of these two intervals. Thus the increase of velocity during the time t is $(l_2 - l_1)/t$ and the acceleration if uniform is $(l_2 - l_1)/t^2$. By making the calculation for various parts of the trace it is found that the same value is obtained from all for the acceleration, and this value is approximately 981 cm. per sec. per sec. It is thus proved that a falling body moves with uniform acceleration.

EXPERIMENT 13. *To shew by observations on falling bodies that in a given locality the acceleration* g *due to gravity is the same for all bodies.*

(*a*) Take two balls of different mass and drop them simultaneously from a height; they will reach the ground at

practically the same moment, they have moved over the same distance in the same time and kept together throughout their course, they have moved with the same velocity, their acceleration has been the same.

An experiment such as the above was first performed by Galileo about 1638. He dropped two shot of different masses from the leaning tower of Pisa and found that they reached the ground together.

The following is a more accurate form of the experiment.

(*b*) Fit two electromagnets on to a wooden frame and connect them so that the same current traverses the two. Pass a current round them and suspend a small iron ball from one, a larger ball from the other; then raise the frame and balls to the top of the room, on breaking the current the balls drop simultaneously and reach the floor together.

Thus two bodies of the same material but of different mass fall at the same rate.

(*c*) Repeat the experiment using for one of the iron balls a wooden ball into which an iron screw or nail has been fixed. By means of this the ball can be held up by the electromagnet. When dropped it will reach the floor very nearly simultaneously with the iron ball.

Hence two balls of different material fall at the same rate.

(*d*) If for the wooden ball a very light ball be substituted it will probably be a little longer in its fall than the iron ball. This is due to the resistance offered by the air to the passage of the balls. In a vacuum the time of fall of any two objects is the same. This can be proved by the aid of a piece of apparatus shewn in Fig. 57. A light and a heavy object—a feather and a sovereign—are placed on two little platforms at the top of a tall glass jar open below. The platforms can be released from outside the jar and the objects allowed to fall. The bottom of the jar is ground

Fig. 57.

and fits in an air-tight manner on the plate of an air-pump. On allowing the two objects to fall when the jar is full of air the heavy one reaches the bottom first. If the air be exhausted and the two then allowed to fall they reach the bottom simultaneously.

66. Acceleration of falling bodies the same for all bodies. We have thus arrived at the important result that in a vacuum all bodies fall to the ground with the same uniform acceleration. This acceleration varies slightly at different places; in England it is approximately 981 cm. per second per second.

In the case of dense bodies the effect of the air is small and we may apply the above results to bodies falling freely through the air.

Hence in a given locality the acceleration with which all heavy bodies fall is the same.

Another and more exact verification of this important law will be given later, see Section 131.

EXPERIMENT 14. *To find a value for* g *the acceleration due to gravity.*

It has been shewn in EXPERIMENT 12 that a falling body moves with uniform acceleration, now in the case of a body so moving we see by putting t equal to unity in the formula $s=\frac{1}{2}at^2$ that *the acceleration is measured by twice the distance which the body describes from rest in the first second,* for we have in that case $a=2s$ when $t=1$.

But it has been shewn in EXPERIMENT 11 that a falling body moves over 490·5 cm. or 16·1 feet in the first second of its fall. Therefore we infer that the value of g is $2\times 490{\cdot}5$ or 981 cm. per second per second. This is equivalent to 32·2 feet per second per second.

67. Weight. A body falls to the earth with uniform acceleration which we denote by g, moreover in a given locality g is constant for all bodies, the mass of the body also is constant, hence the rate of change of the momentum of the body which is measured by mg is constant. But the rate of change of momentum is the **Impressed Force.** Hence the impressed

force is in this case constant. This impressed force is called the **Weight** of the body, if we denote it by W we have the relation $W = Mg$.

Thus in a given locality the weight of a body is constant and in England on the C.G.S. *system it is found in proper units*[1] *by multiplying the mass in grammes by* 981.

Again we have from the above the relation

$$g = \frac{W}{M}.$$

Now in a given locality experiment has shewn that g is the same for all bodies, thus at a given spot the ratio **Weight** to **Mass** is the same for all bodies. *The weight of a body is proportional to its mass;* if we take two lumps of matter one of which has twice the mass of the other the weight of the one will be twice that of the second. We shall see later how this fact is made use of in comparing by weighing the masses of various bodies.

The principle[2] which has just been enunciated that the weight of a body is proportional to its mass is a consequence of the definition of weight and of the experimental fact that in a given locality g is constant for all bodies.

We proceed now to consider some cases of motion in which the moving body is prevented by some means or other from falling freely.

68. Constrained Motion under Gravity.

When a body is allowed to fall freely its velocity soon becomes too great to permit of measurement. Various arrangements have been devised to obviate this. Thus Galileo observed the motion of a ball rolling down a groove in a smooth inclined plane.

He made a series of marks down the groove at distances 1, 4, 9, 16, etc., from the starting point, and found that these marks were passed by the ball at times represented by 1, 2, 3, 4... ; the distance traversed was proportional to the square of the time. Thus the acceleration was constant.

[1] See Section 83. The value 981 varies slightly in England.

[2] On account of the importance of the principle further and more exact experimental means of verifying it will be given in Section 131.

The same end, the reduction of the velocity to a measurable amount, is attained by the use of

Atwood's Machine.

In this apparatus two equal[1] masses P, Q, Fig. 58, are suspended over a light pulley A by means of a fine string: the pulley is mounted so as to experience very little friction and should be as light as is consistent with strength, the masses should be considerable.

This system will remain at rest. A small body called a rider shewn on a larger scale at D in Fig. 58 is placed on the mass P, which commences to descend and the acceleration with which it moves can be observed.

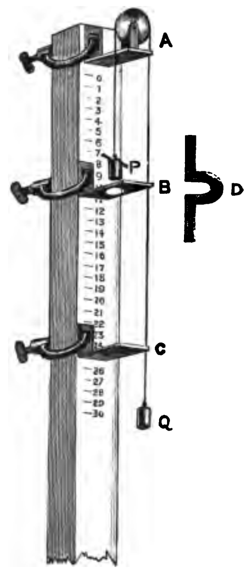

Fig. 58.

The pulley A is carried on a graduated vertical support ABC. At B is a platform which can be clamped at any position on the support. This platform has a circular hole in its centre, through which the mass P can pass but which is not large enough to allow the passage of the rider. C is a second platform which can be placed so as to stop the motion of P at any point of its fall. When the rider has been stopped by the platform B the system continues to move but the velocity becomes uniform. By observing the motion under these circumstances various consequences can be deduced.

In the following experiments we shall describe the observations which can be most conveniently made by a class of students experimenting with Atwood's machine. These will consist in measuring the intervals

[1] These masses may be supposed to be of the same material; they will then be equal if their volumes are equal, if the rider also be of the same material as P and Q its mass may be compared with that of P by a comparison of volumes; we thus avoid the difficulty of comparing the masses of two bodies of different material.

of time taken by the masses in moving over certain measured distances. Each student or small group of students is furnished with a stop-watch, reading to fifths of a second; at a given signal the demonstrator or one of the students releases the masses, the watches are started simultaneously, and stopped as the rider is removed or as the mass P passes some fixed point on the scale as the case may be.

In some Experiments the procedure may be reversed and the distance measured which the masses describe in a given time. When this is done the pendulum described in Section 65 is often useful; the mass P carrying the rider is supported on a small wooden platform, Fig. 59, hinged in the middle. An iron catch holds this platform in a horizontal position, while

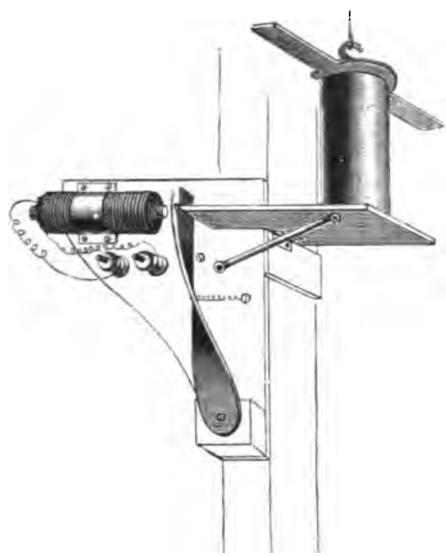

Fig. 59.

a strong elastic band is attached to its under side and pulls it down when the catch is withdrawn. The catch forms the armature of an electromagnet, it is drawn to it each time the magnet is made and is pulled away by a spiral spring when the magnetizing current is broken.

The electric circuit is completed through the spring R of the pendulum (fig. 54) and the connecting screw Q which is connected with

the spring once a second when the pendulum is at the lowest point of its swing. To work the apparatus the pendulum is drawn aside and held fast, the platform is raised and secured by the catch and the rider adjusted. The pendulum is started; as it passes for the first time through its lowest position the current is made and the catch withdrawn with a click; the platform falls and the masses begin to move, the pendulum continues to oscillate and at each second—or half-second if the complete period of the pendulum be one second—a tick is heard as the catch is drawn up to the electromagnet. The platform B can then be adjusted so that the rider is stopped after the motion has continued 1, 2, 3, etc. seconds, and the relation between the distance passed over and the time determined.

Experiment 15. *To investigate the motion in Atwood's machine*[1] *after the rider has been removed and to shew that in this case* $s = vt$.

Place the ring which catches the rider about 25 cm. below the pulley and make marks on the support at distances of 50, 100, 150 cm. etc. below the ring. Take a stop-watch reading to ·2 of a second, raise the mass which carries the small rider to the top and loose the string. Start the stop-watch as the rider is removed and stop it again as the top of the mass passes the first mark below. This will give the time of passing over the first 50 cm. Repeat the experiment but allow the mass to reach the second mark, 100 cm. from the ring, before stopping the watch. It will be found that this second interval is approximately double of the first. Repeat the experiment using the third mark, and so on. We shall thus prove that the spaces moved through after the rider is removed are proportional to the times of motion. The ratio of the space traversed to the time of traversing it measures the velocity and it is thus seen that the velocity after the rider is removed is constant.

The friction of the pulley will usually be sufficient to produce appreciable error in this result. The effects of the friction may be eliminated by making the mass P slightly greater than Q, adjusting the difference so as to counterbalance the friction and to give to the masses when the rider is removed a uniform velocity.

[1] In Experiments 15—17 the masses P and Q may conveniently be each 1000 grammes and the mass of the rider 10 grammes. A piece of waterproofed fishing line makes the best string.

EXPERIMENT 16. *To shew that the velocity produced up to the instant at which the rider is removed is proportional to the time during which the rider has been on.*

In other words, to prove that the acceleration is constant and that $v = at$.

Place the ring of the Atwood's machine so as to remove the rider after the masses have been moving for some definite time (say 4 seconds) as measured by the stop-watch. Having done this, stop the watch, raise the mass and release it, starting the watch as the rider is removed, and as before measure the time occupied by the mass in descending 100 cm. Now lower the ring until the time taken by the mass in descending from rest to the ring is twice that previously occupied, then observe as above the time taken in falling to a point 100 cm. below the ring. It will be found to be half of that taken in the first case. Thus the velocity in the second case is double of that in the first, and the weight of the rider has been impressed on the masses for twice as long. Hence the velocity produced in a given time is proportional to the time. The ratio therefore of the velocity to the time during which it has been produced is constant and this constant ratio is the acceleration. Thus the formula $v = at$ applies to the velocity generated up to the moment at which the rider is removed.

Thus in Atwood's machine, when the rider is on, the masses move with constant acceleration, after the rider is removed they move with constant velocity.

EXPERIMENT 17. *To verify by means of Atwood's machine the formula* $s=\frac{1}{2}at^2$ *where* s *is the space traversed in* t *seconds by a body moving with constant acceleration* a.

We have just seen that when the rider is on the masses move with constant acceleration. Adjust the ring as in the first part of EXPERIMENT 16 so as to find the space passed over in 4, 6, 8, ... seconds. Measure the spaces in each case from the point at which the mass is released; they will be found to be in the ratio of 16 to 36 to 64 or of 4^2 to 6^2 to 8^2. Thus the spaces are proportional to the squares of the time. Now according to the formula $a = 2s/t^2$, thus the acceleration is given by multiplying the space by 2 and dividing by the square of

the time. Form a table of the values of $2s/t^2$ obtained from each experiment; they will be found to be the same within experimental errors and will be equal to the value of the acceleration given by EXPERIMENT 16.

Thus for uniformly accelerated motion we have $s=\frac{1}{2}at^2$.

With the masses mentioned in the foot-note to Exp. 15, if the pulley is light and the friction small it will be found that the distances traversed in 4, 6, 8 ... seconds will be about 39·2, $39 \cdot 2 \times \frac{9}{4}$ and $39 \cdot 2 \times 4$ cm. respectively, and the acceleration about 4·9 cm. per sec. per sec. Thus in the first part of Experiment 16, the ring B will be about 39·2 cm. below the point of release, while after the rider is removed the masses will move with a velocity of 19·6 cm. per second, and the 100 cm. will be described in about 5·1 seconds. In the second part, the ring B will be about 156·8 cm. below the point of release, and the space of 100 cm. will be described in about 2·5 seconds.

69. Rate of change of Momentum in Experiments with Atwood's Machine. In the above experiments the mass moved and the rider remain unchanged; we find then that the acceleration remains constant. The velocity and the space traversed depend on the time during which the system has been in motion; the acceleration does not, it is uniform throughout. Thus we observe that when the rider is unchanged and the mass moved remains constant then the acceleration is constant. Now let the mass of each of the bodies P and Q be M grammes, let m grammes be the mass of the rider. Then the mass moved is $2M+m$ and the rate of change of momentum therefore is $(2M+m)\,a$.

This then is the **Impressed Force.** If the values of the quantities be substituted in this expression it will be found that the result is equal to mg the weight of the rider. Hence *in Atwood's machine the weight of the rider is the impressed force*, and we have $(2M+m)\,a=mg$.

With the numbers given in Experiment 15,

$$2M+m=2010 \text{ grammes},$$
$$a=4\cdot 9 \text{ cm. per sec. per sec.}$$

Thus $\qquad (2M+m)\,a=9849;$

and since $\qquad m=10, \text{ and } g=981,\ mg=9810.$

Hence $(2M+m)\,a$ or the impressed force as given by the definition is shewn by experiment to be very nearly equal to the weight of the rider.

70. Further experiments with Atwood's Machine. We will now examine the effect of varying the mass moved. Take two masses P_1, Q_1 each half as great as P or Q.

In this case the masses moved are P_1 and Q_1 together with the pulley and the rider. Now the pulley and the rider being both of small mass we may say that the mass moved is very approximately[1] half what it was before.

EXPERIMENT 18. *To shew that in Atwood's machine the acceleration produced by the action of a given rider is inversely proportional to the mass moved.*

Replace the masses P, Q on the Atwood's machine by P_1, Q_1 and determine as in Experiment 17, the acceleration by finding the time required to drop some measured distance. Calculate the acceleration a_1 from the formula $a_1 = 2s/t_1^2$, t_1 being the time taken to traverse a distance s. Then it will be found that a_1 is twice a; by halving the mass moved the acceleration is doubled. Thus for example adjust the platform B so that with the large masses P, Q the motion may continue for 10 seconds before the rider is removed. Repeat the observation using the smaller masses P_1, Q_1. It will be found that the distance is now traversed in about 7 seconds. But the accelerations are inversely proportional to the squares of the times in which a given space is described.

Hence $$\frac{a_1}{a} = \frac{10^2}{7^2} = \frac{100}{49} = 2 \text{ approximately.}$$

If follows therefore from this experiment that if the rider be unchanged the acceleration produced is inversely proportional to the mass moved. The product of the mass and the acceleration is constant.

Thus in Atwood's machine when the weight of the rider is constant the product of the mass moved and the acceleration produced is constant.

We will now consider the effect of varying the weight of the rider.

[1] If we wish we can make it exactly half by making P_1 and Q_1 each slightly less than $\frac{1}{2}P$.

EXPERIMENT 19. *To shew that in Atwood's machine the product of the mass moved and the acceleration produced is proportional to the weight of the rider.*

Take a second rider identical with the first; its weight and its mass are the same as those of the first. On placing the two riders on the mass P the weight of the rider is double as great as previously. The mass moved is slightly greater, being increased by the mass of the rider, we may neglect this and say that the masses are approximately the same as in EXPERIMENT 15[1].

Determine now the acceleration by observing the time taken to traverse some distance. The acceleration will be found to be twice what it was previously.

Thus set the ring B so that the masses may move for 10 seconds before the rider is removed. Repeat the experiment using the two riders; then it will be found that this same distance is traversed in 7 seconds. Hence as in EXPERIMENT 18 the acceleration in the second case is twice that observed in the first case; if three riders be used it will be found that the acceleration is trebled.

Thus by combining this result with that obtained in EXPERIMENT 18 we see that *the product of the mass and the acceleration is proportional to the weight of the rider.*

The weight of the rider may be measured by the product of the mass moved and the acceleration produced.

71. Deductions from Experiments on Falling Bodies. There are three points of importance to be noticed which are common to the above experiments:—(1) in all of them the moving system gains momentum; for as the velocity increases with the time, the mass moved remaining the same, the momentum increases also,—(2) in all of them, so long

[1] If we desire to be more accurate we may allow for the slight change in mass by aid of the result obtained in Experiment 18, or we may use two riders one of which is double the other. In the first part of the experiment place the heavier rider on P_1, the lighter on Q_1, the weight producing motion is the difference between the weights of the riders; in the second part place both riders on P_1, the weight producing motion is the sum of the weights of the two, i.e. three times what it was previously.

as the rider remains unchanged, the rate of change of momentum is constant: (3) when the mass of the rider is varied the rate of change of momentum alters in value, remaining constant for experiments with the new rider. Thus momentum is being transferred to the system and, if we know the rate at which this transference is taking place and the original velocity, we can determine the motion at any time.

Newton realized the importance of this quantity—Ma the rate of change of momentum of a moving body—and called it, as we have done, impressed force. Now we have seen from the Definition of Section 63 that the impressed force remains constant so long as the rate of change of momentum remains unchanged. Thus for example the acceleration g of a falling body of constant mass has been proved to be constant, we infer therefore that the force impressed on the body is constant and is equal to Mg.

Now, according to Newton, this impressed force arises from an attraction between the particles of the Earth and those of the falling body, and it is this attraction which is measured by mg and is said to cause the fall of the body.

We cannot strictly prove the existence of this attraction as a *cause* of the motion, all we are really justified in saying is that all the circumstances of the motion not only in the case of falling bodies but in many other cases are consistent with results deduced from the supposition that there is an attraction between the particles of a body and those of the Earth which is measured by Mg.

Thus to take another example we have seen that in the experiments with Atwood's machine, when the weight of the rider is unchanged, the product of the whole mass moved and the acceleration is constant and equal to the weight of the rider. Hence the weight of the rider is the impressed force, this weight is supposed, as in the case of a body falling freely, to arise from the attraction between the Earth and the rider, and to be the cause of the motion.

Hence starting from Force defined as the rate of change of momentum we are led to the conception of some mutual action between bodies measured by this quantity and changing the motion. We must however remember that when we state that the force acting on a body is F, all we know is that momentum is being transferred to the body at the rate of F units per second.

72. Force and Impulse. We can now consider the connexion between Force and Impulse. The impulse measures the whole amount of momentum transferred to the body in

an interval of time; the force measures the rate at which the momentum is transferred.

If during a time t the rate of transference of momentum is constant and equal to F while the impulse or whole amount transferred is I, then the rate of transference is found by dividing the whole change by the time during which it has occurred.

Hence we have

$$F = \frac{I}{t},$$

or

$$I = Ft.$$

When the force is variable this relation still holds provided that the interval t be so small that we may, for that interval, treat the force as constant.

73. Theoretical Mechanics.

In chapters IV. and V. we have discussed some experiments involving simple cases of motion, we have learnt how masses may be compared and have been led to realize the importance of the ideas of momentum and its rate of change to which the name of force has been given.

We are now about to make a fresh start and consider Dynamics as an abstract Science based on certain laws or axioms which were first clearly enunciated by Newton and are called Newton's Laws of Motion. We shall endeavour in the next chapter to explain these laws and to shew how they may be illustrated by the simple cases of motion already discussed; we then go on to assume them as true always and to deduce their consequences in other cases.

We shall not now discuss the question whether these fundamental principles were stated in their best form by Newton. Our present object is to give a consistent account of the Science of Mechanics as it has been developed from Newton's Laws.

CHAPTER VI.

NEWTON'S LAWS OF MOTION.

74. Galileo's Achievements. Galileo investigated the motion of falling bodies, asking the question, How do heavy bodies fall? He shewed that they move with uniform acceleration, which is in a given locality the same for all bodies; he also determined by experiment the relations given by the formulæ

$$v = at, \quad s = \tfrac{1}{2}at^2.$$

Again, calling the weight of a body, which before his time was recognized by the pressure it produced on the hand or table which supported it, **Force,** Galileo shewed that for falling bodies a force could be measured by the acceleration it produced in a given body.

Newton in his Laws of Motion generalized this idea of force as measured by the rate of change of momentum so as to include all cases of motion.

75. Newton's Definition of Force. This is given in the *Principia* as the fourth definition thus: Vis Impressa est actio in corpus exercita, ad mutandum ejus statum vel quiescendi vel movendi uniformiter in directum.

Impressed force is action exercised on a body so as to change its state of rest or of uniform motion in a straight line.

It should be noticed that this definition does not define the measure of a force; it merely states that action exercised so as to change ("to the changing of") a body's state of rest or uniform motion is Force.

In his further definitions Newton calls attention to the fact that the term force as used in his day was measured in various ways. In the Second Law of Motion he states the meaning with which he uses the term and to which throughout the rest of the *Principia* he adheres. Force is now measured in the manner defined by Newton.

76. Newton's Laws of Motion. The definitions of the *Principia* are followed by three Axioms or Laws of Motion[1]. These are given below, and each will be discussed in turn.

LAW I. *Every body perseveres in its state of rest or of uniform motion in a straight line unless it be compelled to change that state by impressed forces.*

LAW II. *Change of motion is proportional to the impressed force and takes place in the direction in which the force is impressed.*

LAW III. *To every action there is always an opposite and equal reaction, or the mutual actions of two bodies are always equal and opposite.*

77. The First Law of Motion. The first point to notice about this law is that it includes the definition of force. For the definition states that force is action exercised on a body to change its state of rest or motion, whilst, according to the law, the state of rest or motion will not change unless force be exerted. So far the two are the same: the law however states more than this; it defines the state of motion in which a body will persevere unless there is impressed force. If in motion, the body will continue to move with uniform speed in a straight line; if the speed alters or the direction of motion changes, force is said to act on the moving body; if at rest, the body will continue at rest unless

[1] In the original Latin the laws are

Lex I. Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum; nisi quatenus a viribus impressis cogitur statum illum mutare.

Lex II. Mutationem motus proportionalem esse vi motrici impressæ, et fieri secundum lineam rectam qua vis illa imprimitur.

Lex III. Actioni contrariam semper et æqualem esse reactionem; sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.

acted on by force. Now the action which Newton calls force arises from the presence of matter; and there is reason for believing that every particle of matter in the universe is influenced in its motion by every other particle; when however the distance between the particles is great, the action between them is extremely small. As it is impossible therefore to have a body entirely free from the action of other bodies it is impossible for us to verify by experiment the first law of motion. The law however asserts that if we could completely isolate a body from the influence of all other bodies it would remain at rest or move with constant speed in a straight line,—the motion of a body will not vary unless influenced by other bodies.

This characteristic of bodies is called Inertia, the first law states the principle of **Inertia.**

At the same time that matter, as inert, is incapable of self-acceleration, incapable that is of changing the speed or direction of its movement, facts justify us in assuming that the motions of any (and every) two particles are mutually affected by each other's presence, the mutual effect in all cases diminishing as the distance between the particles is increased.

Now, though we cannot prove the first law by direct experiment (Newton states it as an Axiom), we can shew that it is consistent with observation; the nearer we approach to the circumstances under which the law is stated to be true the more nearly do observations agree with the results which should follow from the law. A stone will slide further on ice than on a rough road; the friction between the stone and the ice is less than that between the stone and the road, and it is this friction which stops the motion. A lump of iron or lead resting on the ground will not move of itself. Action from some other portion of matter is needed to start it. A body set in motion tends to continue moving.

A ball dropped from the mast-head of a moving vessel strikes the deck at the foot of the mast. The ball at the moment at which it is dropped is moving forwards with the velocity of the ship. This velocity continues during the fall; the ball acquires as well a vertical velocity, it moves with uniform acceleration towards the Earth; but the horizontal velocity remains the same as that of the ship; hence it falls at

the foot of the mast. If a horse suddenly stop there is a tendency for its rider to be pitched forward over its head. The passengers on the back seats of a railway-carriage which is stopped by a collision are thrown forwards against the front of the carriage. A man who steps backwards off an omnibus tends as soon as his feet touch the ground to fall forward on his face; the upper part of his body continues to move with the velocity of the 'bus, his feet are stopped by contact with the ground. In all these cases we have examples of the tendency of motion to continue.

Thus while an appeal to our experience shews us that the law does not contradict observation we cannot thereby prove the law. Our belief in it and in the other laws of motion is really founded on a complicated chain of reasoning. Assuming the laws to be true we can solve the various complicated problems of mechanics; if we find that the solutions which we obtain agree in all cases with observation, and that the agreement is more complete the more completely we apply the laws, we may infer without error that the fundamental principles from which we start are true. Newton applied the laws in combination with his law of gravitation to Astronomy, and shewed how the motions of the planets could be determined, how the eclipses of the Sun might be foretold, and the place from which they would be visible fixed by calculation.

The exact accordance of observation with prediction justifies us in accepting the Laws of Motion as fundamental truths. Mechanics has become a deductive science based on certain definitions and axioms. Experiment and the observation of certain simple cases of motion led Galileo and Newton to recognize certain principles as fundamental. In the Laws of Motion, Newton generalized these principles and applied them to all cases of motion.

78. The Second Law of Motion. *Change of motion is proportional to the impressed force, and takes place in the direction in which the force is impressed.*

In the second definition prefixed to the laws, Newton states that motion, as the term is used in the law, is to be measured *by the product of the mass and the velocity*; it is therefore the

same as the quantity which we have defined as Momentum, while in the eighth definition he explains that Force is measured by the change of Motion (i.e. Momentum), which it produces in *a given time*. Thus by change of motion we are to understand change of momentum occurring in a given time; it is most convenient to take this time as the unit of time, one second; and then *change of motion* as used by Newton becomes in modern language, *Rate of change of Momentum*. We may thus re-state the second law:

LAW II. *Rate of change of Momentum is proportional to the impressed Force and takes place in the direction in which the force is impressed.*

In other words, the law asserts that, when the momentum of a body varies, the rate at which the momentum is changing measures the Impressed Force completely, both in magnitude and direction; this result we have already arrived at in Section 63. Taken in connexion with the definition of Force the law tells us further that we are to take the rate of change of momentum as a measure of action exercised on the body so as to change its motion.

In most cases which occur in Nature the rate of change of momentum of a moving body depends on the position of the body with reference to other bodies. If then we know the position of the body, it is possible to say at what rate its momentum is being changed. Thus if we have two isolated particles free from all external action, each has an acceleration towards the other which is proportional to the mass of the other particle and inversely proportional to the square of the distance between them, the rate of change of momentum is the same for the two; it is equal to mm'/r^2, where m, m' are the masses of the particles and r their distance apart. This quantity which depends only on the distance between the two particles and on their masses is called the force between them. We can calculate the force impressed on a particle, in this case, without knowing how it is moving or what its velocity is. Assuming the mass of the particle to be constant we are thus given its acceleration or the rate at which its velocity is changing in all positions.

From this, if we know the velocity with which the particle starts, it is possible to determine by mathematical reasoning its path and its velocity at any future instant. We have one simple case of this in the problem of a falling body: the acceleration is constant, the velocity is therefore given by $v=u+at$, the space traversed by the formula $s=ut+\frac{1}{2}at^2$.

The fact that in many cases the rate of change of momentum—the force—is constant or depends only on the position of the body is what gives "Force" its importance in Mechanics. We need not look upon it as some external agent causing the motion, it is sufficient for us to know that for each position of a moving body its acceleration is definite, both in magnitude and direction. It does of course happen that in some cases the acceleration depends on the velocity of the particle, such problems are more difficult to deal with. When we say that the force acting on a certain body is X, all we know is that this body is gaining X units of momentum per second.

79. Measurement of Force. Force like other quantities in Mechanics is measured in terms of a unit of its own kind. The second law does not define this unit, for it merely says that Force is proportional to the rate of change of momentum; it would obviously be convenient if we could say that force is *equal* to the rate of change of momentum, and this statement will afford a definition of the unit force, for suppose the momentum of a body to be changing in each second by unity, then the impressed force is unity. We have thus the following definition of unit force:

DEFINITION. *When the momentum of a body changes in each second by unity the impressed force is the* **Unit of Force.**

Hence when the change in momentum per second is 2 units of momentum the impressed force is 2 units of force, and when it is stated that the impressed force is F it is implied that the momentum increases by F units per second.

PROPOSITION 21. *To obtain an equation connecting the rate of change of momentum of a body and the impressed force.*

Let u be the initial velocity of a particle of mass m, v its velocity at the end of t seconds, a its acceleration, and F the impressed force. We suppose F to be constant during the time.

Then in one second the particle gains F units of momentum. But the original momentum was mu, the momentum after t'' is mv.

Hence change of momentum in t seconds is $mv - mu$. Therefore change of momentum in 1 second is $(mv - mu)/t$.

Therefore $$F = \frac{mv - mu}{t} = \frac{m(v-u)}{t}.$$

But $$v = u + at.$$

Hence $$\frac{v-u}{t} = a.$$

Therefore $$F = ma.$$

Thus the product of the mass and the acceleration is equal to the impressed force provided that F, m and a are measured in a consistent system of units in which the impressed force is unity, when the gain of momentum per second is unity.

80. The C.G.S. Unit of Force. According to the C.G.S. system the unit of mass is 1 gramme, the unit of velocity is a velocity of 1 centimetre per second. Hence if the velocity of a mass of 1 gramme increases per second by 1 cm. per second the mass gains momentum at unit rate. The impressed force therefore is the C.G.S. unit of force. The C.G.S. unit of force is called a Dyne.

DEFINITION OF ONE DYNE. *When a mass of one gramme gains per second a velocity of one centimetre per second, the impressed force is one* **Dyne.**

Hence if a mass of m grammes has an acceleration of a centimetres per second per second, the impressed force is ma **Dynes.**

Or again, if we know that the impressed force is F dynes, and the mass m grammes, then the acceleration a in centimetres per second per second, is given by the equation

$$F = ma,$$

or $$a = \frac{F}{m}.$$

81. The F.P.S. Unit Force. On the English or F.P.S. system of units the unit mass is 1 pound; the unit of

velocity is a velocity of 1 foot per second. Thus if the velocity of a mass of 1 pound increases per second by a velocity of 1 foot per second, the impressed force is the English or F.P.S. unit force. The F.P.S. unit force is called the Poundal.

DEFINITION OF ONE POUNDAL. *When a mass of one pound gains per second a velocity of one foot per second, the impressed force is one* **Poundal.**

Hence if a mass of m pounds has an acceleration of a feet per second per second, the impressed force is ma **Poundals.**

Examples. (1). *The velocity of a mass of* 10 *grammes is changed in* 5 *seconds from* 25 *to* 125 *cm. per second, find the force.*

The change in velocity in 5 seconds is

$$125 - 25 \text{ or } 100 \text{ cm. per second;}$$

$\therefore$ the acceleration or change in velocity per second is 100/5 or 20 cm. per sec. per sec.

The mass is 10 grammes.

Hence the force is 10×20 or 200 dynes.

(2). *A train whose mass is* 20 *tons moves at the rate of* 60 *miles an hour; after steam is shut off it is brought to rest by the brakes in* 500 *yards. Find the force exerted, assuming it to be uniform.*

[To solve this problem we must express the velocity in feet per second, the mass in lbs., then find the retardation and hence the force.]

A velocity of 60 miles an hour is $\dfrac{1760 \times 3 \times 60}{60 \times 60}$, or 88 feet per second.

Let a be the retardation; this velocity is destroyed in a space of 500 yds.

Hence applying the formula $v^2 = 2as$, we have

$$2 \times a \times 500 \times 3 = 88^2 = 7744;$$

$$\therefore a = 2{\cdot}5813 \text{ feet per sec. per sec.}$$

Now the mass of the train is

$$20 \times 20 \times 112 \text{ or } 44800 \text{ pounds.}$$

Hence the

$$\text{Force} = 44800 \times 2{\cdot}5813 \text{ poundals}$$
$$= 115642 \text{ poundals.}$$

(3). *Find the force if in* 1 *second a mass of* 1 *gramme gains a velocity of* 981 *cm. per second.*

The acceleration is 981 cm. per sec. per sec., and the mass is 1 gramme. Thus the force is 1×981 or 981 dynes.

(4). *How many dynes are there in a poundal?*

A poundal is the impressed force when a mass of 1 lb. has an acceleration of 1 ft. per second per second.

Now 1 foot contains 30·48 cm., and 1 pound contains 453·6 grammes.

Therefore if the impressed force be 1 poundal a mass of 453·6 grammes has an acceleration of 30·48 cm. per sec. per sec.

But the number of dynes equivalent to this is $453·6 \times 30·48$ or 13826 —omitting decimals;

$$\therefore \text{ 1 Poundal} = 13826 \text{ Dynes.}$$

Thus we see that a Dyne is a very small force compared with a Poundal.

82. Comparison of Forces. Forces are measured by the momenta which they communicate per second to any mass. If we restrict ourselves to one body the momentum gained per second by that body will be proportional to its acceleration. Two forces then can be compared by comparing the accelerations communicated by them to the same body.

Thus, for example, place a rider on one of the suspended masses on an Atwood's machine and observe the acceleration; change the rider and again observe the acceleration; the weights of the two riders are proportional to the two accelerations [1].

Or, again, it is found by experiment that near the Equator a body falls with an acceleration of about 978 cm. per sec. per sec., while in high latitudes near the Pole the value of this acceleration is about 983 cm. per sec. per sec. Thus the weight of a body is greater near the Pole than near the Equator in the ratio of 983 to 978; in going from the Equator to the Pole a body gains in weight about 5 parts in 1000. (See Section 134). Observations on the Moon shew that it has an acceleration towards the Earth of about ·27 cm. per second per second. The acceleration towards the Earth of any body when close to the Earth is about 980 cm. per second per second. Thus if the Moon were close to the Earth its weight would be increased in the ratio of 980 to ·27 or about 3600 times.

[1] It is assumed here that the mass of the rider is small compared with the suspended masses.

DEFINITION OF EQUAL FORCES. *Two forces are said to be equal when the velocities which they communicate per second to the same mass are equal.*

83. Comparison of Masses. We have seen already (§ 51) how masses may be compared by Prof. Hicks' ballistic balance, and from experiments with it we have obtained an idea of the meaning of the term mass.

The second law of motion gives us another method of comparing masses consistent with the above.

For let M_1, M_2 be two masses, a_1, a_2 the accelerations communicated to them by a given force F. Then by the second law

$$M_1 a_1 = F = M_2 a_2.$$

Thus
$$\frac{M_1}{M_2} = \frac{a_2}{a_1}.$$

Thus two masses are inversely proportional to the velocities which are communicated to them per second by the same force.

DEFINITION OF EQUAL MASSES. *Two masses are equal when a given force communicates*[1] *to them per second the same velocity.*

The simplest method theoretically in which we could apply this method of comparing Masses would be to imagine the two masses free from external action and capable of motion under their mutual action only; the accelerations of each body ought then to be observed and would be proportional to the mass of the other body. But in practice any such method is impossible. Now, we have learnt from experiments with Atwood's machine that in a given locality the weight of a body is a constant force. We may use, then, the following method for comparing masses. Place two equal masses on an Atwood's machine and observe the acceleration communicated by a given rider. Replace these masses by two other equal masses and again

[1] We may put this otherwise thus. When two bodies are gaining momentum at the same rate the force acting on each of them is the same. If they are also gaining velocity at the same rate the masses of the two are equal.

observe the acceleration. The masses suspended in the two cases are inversely proportional to the accelerations.

We may sum up the conclusions of the last two sections thus:

By considering the accelerations communicated to the *same mass by different forces*, we see that

(1) Two forces are equal when they communicate the same acceleration to a given mass.

(2) A force is proportional to the acceleration it communicates to a given mass.

Whilst by considering the acceleration communicated to *different masses* by the *same force*, we see that

(1) Two masses are equal when a given force communicates the same acceleration to each.

(2) The mass of a body is inversely proportional to the acceleration communicated to it by a given force.

84. Falling Bodies and the Second Law of Motion. When a body falls freely it moves downwards with acceleration g; let M be its mass, the impressed force is the weight of the body, let it be W. Then since W is the force which communicates to the mass M its acceleration g we have the relation

$$W = Mg.$$

The acceleration g is constant for all bodies in a given locality, but varies from place to place.

In obtaining this result we have assumed that our measurements are made in a consistent system of units. The force must be measured in units such that the unit force communicates per unit of time to the unit of mass unit velocity; if we work on the C.G.S. system the force is measured in Dynes. If we work on the F.P.S. system it is measured in Poundals.

85. Value of a Dyne. We can use this result to enable us to specify in a more concrete form what a force of 1 dyne is.

For consider a mass of 1 gramme so that M is 1 in the formula and W stands for the weight of a mass of 1 gramme.

We have

$$W = g,$$

or *the weight of a mass of* 1 *gramme contains* g *dynes.*

Now, as we have said, the value of g varies at different parts of the Earth. In England we may take it as equal to 981 cm. per sec. per sec. We learn therefore

(1) That the weight of a mass of 1 gramme varies at different parts of the Earth.

(2) That the weight of a mass of 1 gramme in England contains 981 dynes.

Thus 1 dyne is $\frac{1}{981}$ of the weight of 1 gramme in England.

Or, in other words, divide a mass of 1 gramme into 981 parts. The weight of each part in England is 1 dyne. Since 981 is not very different from 1000 we may say that roughly a dyne is one-thousandth part of the weight of 1 gramme, or is equal to the weight of a milligramme, so that if we apply to a mass of 1 gramme a force equal to the weight of 1 milligramme it will acquire approximately an acceleration of 1 cm. per sec. per sec. (The real value of the acceleration will be ·981 cm. per sec. per sec.)

86. Value of a Poundal. The equation $W = Mg$ will apply equally well to the F.P.S. system. In this case M is in lb., W in poundals, g in feet per sec. per sec., and we may take approximately $g = 32$ feet per sec. per sec.

Consider now the case in which the mass is 1 lb. Then $M = 1$, and W is the weight of a mass of 1 lb.

Thus $W = g = 32$ approximately, and we have the result that

The weight of a mass of 1 pound contains 32 poundals, or

1 Poundal $= \frac{1}{32}$ of the weight of 1 lb. = the weight of half an ounce.

Thus if we apply a force equal to the weight of half an ounce to the mass of 1 pound the mass will acquire approximately an acceleration of 1 foot per second per second. (The result is only approximate, because g is not accurately equal to 32 feet per sec. per sec.)

87. Relation between Weight and Mass. Since at any point on the Earth's surface the acceleration of a falling body is the same for all bodies, it follows that the weights of two bodies are proportional to their masses, for the ratio of the weight to the mass measures in each case the acceleration produced, and this is the same for the two. Thus at any given point on the Earth the weight of a body is proportional to its mass.

Since the acceleration of a falling body is different at different points on the Earth the ratio of the weight of a given body to the mass of that body differs from point to point; the mass of the body is the same everywhere, hence we infer that the weight of a body differs from point to point. The mass of a body is an invariable quantity; the weight of a body depends on its position.

CHAPTER VII.

FORCE AND MOTION.

88. The Action of Force. In Chapter v. Force has been defined as Rate of Change of Momentum. We do not need for the purposes of Mechanics to discuss the question whether there is some cause external to a moving body which acts upon it and makes it move; we can leave the question of efficient causes out of sight. We can observe the velocity and the acceleration of moving bodies, we find in many cases that the product of the mass and the acceleration does not depend on the motion of the body, but is either constant or depends on its position relative to surrounding bodies. This it is true lends plausibility to the idea that this quantity—the impressed force as it has been called—is something external to the body efficient in making it move; thus the phrases "the forces *acting* on the body," "the forces *producing* motion," and the like are in common use.

We have however no right to say that force *produces* motion, or that force *acts* on a body, if we attach to the words 'produce' and 'act' their ordinary meaning, implying the existence of some agent or cause to which the motion can be assigned and define force as above. At the same time we may conveniently use the phrase "the forces acting on the particle" and the like if we do it in a sense limited by our own definition. *All that we mean by the statement that a force is acting on a body is that the momentum of that body is changing.* We define the *acting* of force thus :

DEFINITION. *When the momentum of a body is changing gradually, force is said to act on the body.*

Again, a body may be in equilibrium under the action of several forces; in this case the accelerations which the body would have were each force to act singly are so related that their resultant is zero.

The relation between acceleration and the corresponding force is always given by the equation,

$$F = ma.$$

89. Law of Gravitation. The change of motion of any particle depends on its position with regard to other bodies.

Each particle in the universe has an acceleration towards all the other particles; the acceleration which we observe is the resultant of these innumerable component accelerations.

If we have two particles A and A_1 of masses m and m_1 at a distance r apart, A has, as we have already said, an acceleration towards A_1, and its amount is m_1/r^2, A_1 has an acceleration towards A, and its amount is m/r^2; the impressed force on A is therefore mm_1/r^2 towards A_1, while that on A_1 is m_1m/r^2 in the direction A_1A. These two forces are equal and opposite; we may express this fact by saying that there is an attraction mm_1/r^2 between the particles A and A_1.

Newton, in the law of gravitation, asserts that this is true for every pair of particles in the Universe. In the *Principia* he calculated the Motion of the Planets and their Satellites, assuming this law to be true; the fact that motion so calculated agrees with observation justifies his assumption.

The law is usually stated thus:

LAW OF GRAVITATION. *Every particle of matter attracts every other particle with a force which is proportional to the product of their masses and inversely proportional to the square of the distance between them.*

We can put the law rather differently thus, and in this form it represents more accurately the results of observation: *Every particle in the Universe has an acceleration towards every other particle. The amount of the acceleration towards any*

second particle is proportional to the mass of this second particle, and inversely proportional to the square of the distance between the two.

To determine then the acceleration of a particle completely we calculate the resultant of these component accelerations.

Thus any particle near the Earth has accelerations directed to the particles of which the Earth is composed, and also towards those of the Sun, the Moon and the stars. Owing however to the immense distances of these bodies these latter accelerations may be neglected and the effect of the Earth only calculated.

Now the Earth is very nearly a sphere, and Newton shewed that for an external particle the resultant of the acceleration due to the particles of a homogeneous sphere is the same as it would be if the sphere were replaced by a single particle at its centre; the mass of this particle being equal to that of the sphere. It is easy thus to calculate the acceleration of a particle free to move near the Earth; if M be the mass of the Earth, and R its radius, the value of this acceleration will be that due to a mass M at a distance R; its amount therefore is M/R^2, and its direction will be towards the centre of the Earth.

Thus a particle close[1] to the Earth's surface will have a vertical acceleration M/R^2, and this is the same whatever be the mass of the particle.

The impressed force on the particle will therefore be mM/R^2, and this is w the weight of the particle.

Thus we have

$$w = \frac{mM}{R^2}.$$

Again, we may look upon a body of finite volume as an aggregation of particles. Each of these has the same acceleration M/R^2 towards the Earth; thus the whole body has this

[1] We assume that the particle is so near the Earth's surface that its height h may be neglected compared with R: the true value of the acceleration is $M/(R+h)^2$, and if h is small compared with R (4000 miles) we may call the acceleration M/R^2.

acceleration, and if m now represent the mass of the whole body, and W its weight, we have

$$W = \frac{mM}{R^2}.$$

Thus the weight of a body is proportional to its mass; the acceleration with which it falls is equal to the mass of the Earth divided by the square of its radius and does not at all depend on the body.

On the other hand, the Earth is not accurately a sphere, its polar diameter is less than its equatorial diameter, and though we may, without serious error, calculate the acceleration of the particle as though the Earth were spherical, and suppose the whole mass concentrated at the centre we must remember that the quantity R, the distance between the Earth's centre and the point for which our calculations are made, differs for different points on the Earth, being less near the poles than near the equator.

For this reason the acceleration is greater near the poles than near the equator.

The motion of the Earth round its axis tends also to reduce the acceleration as the particle approaches the equator (see § 143).

Thus Newton's law of gravitation leads to conclusions in accordance with those deduced from the second law of motion as to the relation between mass and weight. For a definite position on the Earth the weights of all bodies are proportional to their respective masses; at different points on the Earth the same body has different weights.

In the foregoing section we have stated that the acceleration of a mass m' when at a distance of r centimetres from a second mass m is m/r^2, and that the force is mm'/r^2. But the acceleration so measured is not given in centimetres per second per second; if it were, then if we placed two small bodies each 1 gramme in mass at a distance of 1 centimetre apart, they would have an acceleration toward each other of $1/1^2$ or 1 centimetre per second per second.

Now the acceleration of two masses under such circumstances has been determined by means of the torsion balance and in other ways, and

has been found to be very much smaller than 1 cm. per sec. per sec. It is in fact about

$$6 \cdot 698/10^8 \text{ cm. per sec. per sec.}$$

and the impressed force on each mass is

$$6 \cdot 698/10^8 \text{ dynes.}$$

Hence the impressed force on a particle m grammes in mass at a distance of r centimetres from a particle of equal mass is

$$\frac{6 \cdot 698}{10^8} \frac{m . m'}{r^2} \text{ dynes.}$$

90. Gravitational Unit of Force. We can if we like measure all our forces in terms of the weight of some given body, say 1 lb. In such a case of course when speaking of a force P we do not mean a force of P dynes, or P poundals, but a force P times as great as the weight of 1 pound. Such a unit is known as a **gravitational unit of force.** It depends on the attraction between the Earth and a body having a mass of 1 pound. Now this attraction depends on the position of that mass on the Earth; it is greater as said above near the poles than near the equator; thus the gravitational unit of force is different at different points, a force, 10 say, would be really a larger force near the poles than near the equator; it would mean in each case ten times the weight of a certain lump of matter and this force is greater in high latitudes than in low.

If we are working with gravitational units we cannot use the equation $F = ma$, for this supposes that the unit force communicates unit acceleration to the unit of mass; now in gravitational units this supposition is not true, the unit force is the weight of 1 pound, the unit of mass is the mass of 1 lb. and the unit force communicates acceleration g to the unit of mass; the weight of 1 pound contains g poundals, the gravitational unit is g times as great as the absolute unit (§ 79).

We can determine the relation between the acceleration and the force when gravitational units are employed thus.

Let W be the weight of the body in pounds, F the force acting on it in pounds' weight, a the acceleration produced in feet per second per second.

Then we have

W communicates to the given body an acceleration g,

F .... a.

But two forces are respectively proportional to the accelerations they communicate to a given body.

Hence $$F : W = a : g.$$

$$\therefore \frac{F}{W} = \frac{a}{g}.$$

$$\therefore F = \frac{W}{g} . a.$$

Example. (1) Find in gravitational units the force required to give in $\frac{1}{5}$ of a second to a mass of 1 cwt. a velocity of 100 feet per second.

Since in $\frac{1}{5}$ second a velocity of 100 ft. per second is produced, a velocity of 500 feet per second is produced in 1 second.

Hence the acceleration is 500 ft. per sec. per sec.

The weight of 1 cwt. is 112 lb. wt.; thus taking g as 32

$$F = \frac{112}{32} . 500 = 1750 \text{ lb. weight.}$$

Thus at a place at which g is equal to 32, a force of 1750 lb. weight acting on 1 cwt. will in $\frac{1}{5}$ of a second produce a velocity of 100 feet per second.

(2) Find what velocity this force will in $\frac{1}{5}$ of a second produce in a mass of 1 cwt. at a place at which the value of g is 32·2 ft. per sec. per sec.

Let a be the acceleration.

Then $$\frac{a}{32 \cdot 2} = \frac{1750}{112} = \frac{500}{32};$$

$$\therefore a = \frac{32 \cdot 2}{32} \times 500 = 503 \cdot 125 \text{ ft. per sec. per sec.}$$

Hence the velocity produced in $\frac{1}{5}$ sec. is 100·625 feet per second.

Hence when gravitational units are used two forces *nominally* the same produce different effects at different points on the Earth.

91. Equilibrium. When a body is in equilibrium it has no acceleration; the total impressed force therefore is zero; now it is often desirable to look upon this state of no acceleration as the consequence of the superposition of two or more accelerations which are so related that their resultant is zero. When there are only two such accelerations they are clearly equal and opposite.

Thus consider a body supported by a string: were it free it would move to the Earth with uniform acceleration g; its freedom however is modified by its connexion to the string; since it

has no acceleration the action of the string must be such that were the Earth removed the body would begin to move upwards with acceleration g; in this case the impressed force would be upwards and equal to mg. This impressed force is called the tension of the string. The particle is at rest, hence the upward tension of the string is equal to the weight of the particle and the total impressed force is zero.

Again, consider a body attached to a vertical spiral spring, let the body be at first supported in such a position that the spring is unstretched, gradually lower the support, the body falls, and the spring is stretched until the body is left suspended from the spring. In this case also there is action between the spring and the body. There is an impressed force on the spring equal to the weight of the body.

Just then in the same way as we may look upon the actual acceleration of a body as the resultant of a number of component accelerations, so we may consider the impressed force as the resultant of a number of impressed forces. We shall see shortly how the component forces and their resultant are related together.

92. Comparison of Masses by "Weighing."

We have seen that it follows from the definition of force that the weight of a body is proportional to its mass; this result also is in accordance with Newton's law of gravitation. Two bodies then of equal weight are equal in mass; this fact may be made use of as a means for comparing masses. This is the principle of the ordinary balance (see *Statics*, Section 59); the balance enables us to determine when the weights of two bodies are equal. When this is the case we infer that the masses are equal also.

It is this equality of mass which we usually wish to secure in weighing. In buying a pound of tea or a pound of sugar the customer cares nothing about the attraction of the Earth for the tea or sugar. He wishes to know that he is obtaining for his money an amount of tea equal to that which he has been in the habit of receiving for that sum.

This end is secured most readily by comparing in each case the mass of tea purchased with some standard mass, a pound or kilogramme. The comparison by the balance of the weight of the tea and of the standard

pound affords the easiest method of comparing their masses. If he buys two equal masses of tea of the same quality for the same sum of money he infers that the price he is paying per cup for tea of a definite strength remains unchanged, and this is what he wishes to know.

The fact that the same term—pound or kilogramme as the case may be—is used in ordinary language both for mass and weight, is no doubt productive of confusion. A pound is strictly a denomination of mass but it and other names of mass are also used as a denomination of force; thus Engineers speak of a pressure of so many pounds to the square inch or of the breaking stress of a piece of material being so many tons; in such cases the phrase is an abbreviation. A force of P pounds means a force equal to the weight of a mass containing P pounds.

93. Methods of measuring Force. The method of measuring force by the acceleration it communicates to a body is not always the most convenient. For many purposes as we have seen a force may be measured in terms of a weight; a force of a certain amount F acts on a body in a given direction, we can imagine a string attached to the body and passing from it in the given direction over a smooth pulley. Hang a mass whose weight is equal to the force on to the string, then the force acting along the string is represented by the weight.

If the force acts in a vertical direction on a body it may be represented by the weight of a mass placed directly on the body.

Now in some cases force when it acts on a body changes visibly the size or shape of the body. If a body of considerable mass be placed on an indiarubber ball, the ball is squeezed and flattened; if the same mass be suspended by a piece of indiarubber the indiarubber is lengthened; in these cases force acting on the body visibly[1] alters its shape. In many such cases we can shew that the change in shape is proportional to the force.

Thus take a spiral spring[2], fasten one end to a fixed support and suspend a light scale-pan from the other end.

[1] The shape or size of nearly all bodies is altered by the action of force; with many bodies however the alteration is too small to be detected unless special means of observation are employed.

[2] Such a spring is easily made by winding a piece of steel or brass wire about 1 mm. in diameter in a spiral coil on a rod of circular section

Attach a piece of wire to form a horizontal pointer to the lower end of the spiral spring and adjust a scale in a vertical position as shewn in Fig. 60, so as to mark the position of the pointer. Note the position of the pointer on the scale, then place a mass in the scale-pan, the spring is extended. Note the position of the pointer and remove the mass; unless the mass be too large[1] the pointer will be found to return to its original position.

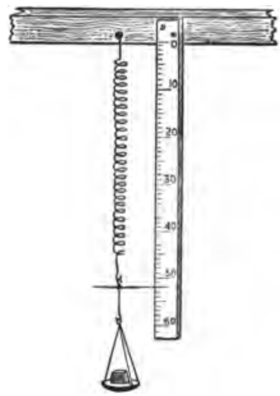

Fig. 60.

On replacing the mass, the spring is again extended and to the same amount as before. When the body is placed in the scale-pan a force, the weight of the body, acts on the scale-pan, the spring is stretched and for a given force the extension is constant. Now replace the body in the scale-pan by one of double the mass and therefore of double the weight. Observe in this case the position of the pointer and again measure the extension. Unless the weight is too large for the spring it will be found that in this case the extension is double that observed previously. The force acting is doubled and the extension also. By varying the mass we may shew that in all cases, so long as we keep within the elastic limits of the spring, the extension is proportional to the force. Thus the extension of the spring may be made use of to measure the force. This is done in the ordinary form of spring balance such as is used

some 2 or 3 cm. in diameter, the rod is placed in a lathe and turned slowly by hand while the wire is wound on to it under considerable tension.

[1] By loading the scale-pan too heavily the spring may be permanently stretched so that when the body is removed the pointer does not come back to its original position: if this is done the spring is said to be stretched beyond its elastic limits and the relation between extension and force no longer holds.

for weighing letters. The balance is graduated by hanging on masses of 1, 2, 3 etc. pounds. When a body of unknown mass is suspended its mass is determined by observing the position on the scale at which the pointer rests.

It should be noticed that a spring balance measures the force applied. If used to determine mass it will only do so correctly in the latitude at which it was graduated. For suppose it graduated in London, suspend a mass of say 1 kilogramme and carry the whole northwards; the mass suspended remains the same, but as the pole is approached the weight of that mass increases, the force acting on the spring becomes greater and the spring is stretched further, the balance therefore reads over 1 kilogramme, but if it is inferred from this that the suspended mass is greater than a kilogramme the inference is wrong. A similar result though in the opposite direction will take place if the balance be carried towards the equator. Thus the extension of a spiral spring affords another method of estimating force.

94. The Composition of Forces. The motion or the equilibrium of a body depends on its relations to other bodies.

Let us suppose that we know the motion which would follow were the body free in turn from the action of all but one of the bodies which can affect its motion. We know then the forces which act separately on the body, and we wish to deduce from this knowledge what will happen when those forces are combined and act simultaneously.

Now the second law of motion states that rate of change of momentum is proportional to the impressed force and *takes place in the direction of that force.* We can extend the application of this law to the case of a number of forces thus. Calculate the acceleration of the body under the action of each force separately, combine the accelerations according to the parallelogram law, then the resultant acceleration is that which the body will have when the combined forces are impressed on it. The acceleration corresponding to a given force is independent of other velocities or accelerations which the body may possess. The action of each force is unimpeded by the others, the observed motion—or rest—is the result of all.

We shall see however in the following sections (§ 96 seq.) that there is a rule by which a number of forces may be com-

bined and their resultant found. This rule was given by Newton as a Corollary to the laws of motion; it is often simpler to proceed by its aid and then to calculate the acceleration under the action of this resultant force rather than to reverse the process and after finding the acceleration corresponding to each force combine these so as to obtain the actual motion.

Thus the second law of motion in the form in which it has been stated involves the principle that,

The effect of a force on a body, as measured by the rate of gain of momentum, is independent of other forces which may be impressed.

This is sometimes spoken of as the independence of forces.

95. Representation of a Force.

Proposition 22. *To prove that forces can be represented by straight lines.*

To define a force we need to know the number of units of force it contains, the direction in which it acts and the point at which it is impressed. These can be represented by a straight line, for a straight line can be drawn from a given point—the point of application—in a given direction—the line of action of the force—and so as to contain a given number of units of length—the number of units of force in the given force. More briefly the proof can be put thus: A force is measured by the acceleration it can communicate to unit mass, and acceleration can be represented by a straight line.

Thus if we suppose that a length of 1 centimetre represents the unit of force then a line AB, 5 cm. long, drawn from A to B, represents a force acting at A in the direction AB and containing 5 units of force.

96. The Parallelogram of Forces.

Proposition 23. *If two forces impressed on a particle be represented in direction and magnitude by two adjacent sides of a parallelogram the resultant of the forces is represented by the diagonal of the parallelogram which passes through their point of intersection.*

Let OA, OB, Fig. 61, represent the two forces.

Complete the parallelogram $AOBC$ and draw the diagonal OC.

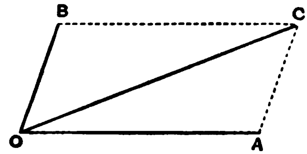

Fig. 61.

A force is measured by the velocity it communicates per second to a particle of unit mass.

Therefore OA, OB represent the velocities which the force would communicate per second to a particle of unit mass.

Therefore, by the parallelogram of velocities, OC would be the resultant velocity of the particle if the forces acted on it for one second. Thus OC represents the force which acting on the particle for a second would communicate to it its actual velocity.

Hence OC represents the resultant force; now OC is the diagonal of a parallelogram whose sides OA, OB represent the forces P, Q respectively. Thus the Proposition is true.

The formulæ and propositions established in §§ 29—32 about the composition and resolution of velocities and displacements will therefore apply to forces. The development of these formulæ applied to bodies at rest gives us the Science of Statics, which is considered with experiments in the second part of this book. The following sections illustrate the resolution of forces and the application of the second law of motion to some simple problems.

97. Problems on Motion.

PROPOSITION 24. *To determine the motion of a body sliding down a smooth*[1] *inclined plane.*

Let m be the mass of the particle, α the angle between the plane and the horizon, R the force between the plane and the particle, a the acceleration of the particle along the plane.

[1] A surface is said to be "smooth" when the direction of the force between it and any body in contact with it is at right angles to the surface.

Resolve the forces along and perpendicular to the plane ABC, Fig. 62, and equate in each case the rate of change of momentum to the force in its direction.

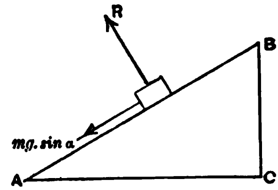

Fig. 62.

The weight of the particle is mg dynes and it acts vertically downwards; this can be resolved into

$$mg \cos a$$

perpendicular to the plane and

$$mg \sin a$$

along the plane. Thus the force perpendicular to the plane is

$$R - mg \cos a.$$

The force along the plane is

$$mg \sin a.$$

There is no acceleration perpendicular to the plane while the acceleration down the plane is a.

Hence

$$0 = R - mg \cos a,$$

$$ma = mg \sin a,$$

$$\therefore \ R = mg \cos a,$$

$$a = g \sin a.$$

Thus the force on the plane is $mg \cos a$ dynes while the particle slides down with uniform acceleration $g \sin a$.

Hence, if l be the length of the plane, t the time taken by the particle in sliding down it, and v the velocity of the particle at the bottom, assuming it to start from rest at the top, then

$$l = \tfrac{1}{2} g \sin a \,.\, t^2,$$

$$v = g \sin a \,.\, t,$$

$$v^2 = 2g \sin a \,.\, l.$$

PROPOSITION 25. *To determine the motion of two particles suspended by a fine string over a smooth pulley.*

Let m and m' be the masses of the particles A and B, Fig. 63, and let m be greater than m', we suppose the mass of the pulley may be neglected. Since m is greater than m', the mass A moves downwards while B moves up; let a be the common acceleration.

The weight of A is mg, that of B is $m'g$, both these forces act downwards but the former acts on A in the direction of its motion, the latter acts on B in a direction opposite to that of its motion. Thus the impressed force in the direction of motion is $(m - m')\,g$.

The mass moved is $m + m'$, hence the rate at which momentum is gained by the system is $(m + m')\,a$.

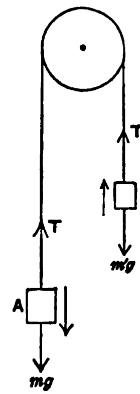

Fig. 63.

Therefore equating this to the force

$$(m + m')\,a = (m - m')\,g.$$

Hence
$$a = \frac{m - m'}{m + m'}\,g.$$

Thus the system moves with a uniform acceleration which is a definite fraction of that due to gravity. The space passed over and the velocity generated in a given time can be found in the usual way.

PROPOSITION 26. *To find the tension of the string joining the two masses suspended over a pulley as in the last Proposition.*

To obtain this we must consider the motion of each mass separately, remembering that since they are connected by the string the upward acceleration of m' and the downward acceleration of m must be the same. Moreover the tension of the string is the same throughout. Let it be T dynes, then a force T acts upwards on both m and m'. Let the acceleration be a.

The weights of m and m' are mg and $m'g$ dynes respectively and act downwards, and we know by the second law that the

product of the mass and the acceleration is equal to the force in the direction of motion.

Hence for the downward motion of m

$$ma = mg - T,$$

while for the upward motion of m'

$$m'a = T - m'g.$$

By adding the two equations we have as in Proposition 25 above

$$(m + m')\,a = (m - m')\,g,$$

$$a = \frac{m - m'}{m + m'}\,g.$$

Multiply the first equation by m', the second by m and subtract, we then find

$$0 = 2mm'g - T(m + m').$$

Hence $$T = \frac{2mm'g}{m + m'}.$$

We have thus found the tension of the string.

The three preceding propositions give us examples of the method to be followed in solving all mechanical problems. We divide the process into three parts: (1) the formation of the equations of motion, (2) the solution of those equations, (3) the physical interpretation of the solution.

Under (1) we express the forces and the rates of change of momentum in terms of symbols and form the equations by equating in accordance with Newton's second law the forces and the corresponding rates of change of momentum; this constitutes the fundamental dynamical part. Under (2) we apply the methods of Algebra or of Trigonometry to the solution of the equations, and obtain the unknown accelerations or forces in terms of known quantities. Under (3) we interpret the solution we have found.

98. Experiments on the value of g. The last propositions furnish us with the theory of experiments which may be used to find g.

EXPERIMENT 20. *To find g by observation on a body sliding down a smooth inclined plane.*

We have seen that if α be the angle of the plane, the acceleration is $g \sin \alpha$ down the plane, and if the time t of moving down a length l of the plane be observed,

$$l = \tfrac{1}{2} g \sin \alpha \,.\, t^2.$$

Hence $$g = \frac{2l}{t^2 \sin \alpha}.$$

Again if h be the height of the plane, l its length, then

$$\sin \alpha = h/l,$$

and $$g = \frac{2l^2}{ht^2}.$$

To make the observation, obtain a sheet of glass or a smooth board with a groove down it about a metre long. Raise one end until it is some 2 or 3 cm. in height above the other, measure this height carefully and also the length of the plane. Place a smooth marble or small ball at the top and observe with a stop-watch the time taken by it in rolling down, then the values of l, h and t are known, and g can be calculated. The results however will not be very accurate because of friction and the difficulty of observing the time accurately.

EXPERIMENT 21. *To determine* g *by means of Atwood's Machine.*

In Atwood's Machine let M grammes be the mass of each of the two large weights, m grammes that of the rider; then omitting the effect due to the mass of the pulley and to friction, the mass moved is $2M + m$, the force producing motion is the weight of the rider or mg dynes. Let the acceleration be a, then the rate of change of momentum is $(2M + m)a$ and we have $$(2M + m)a = mg.$$

Therefore $$g = \frac{2M + m}{m} a.$$

Raise the mass P, Fig. 58, p. 103, and putting on the rider observe as in Experiment 17 the time taken to fall through some convenient distance, say 2 metres; let this distance be s cm. and the time of fall t seconds, then

$$s = \tfrac{1}{2}at^2.$$

Hence $$a = \frac{2s}{t^2},$$

$$g = \frac{2M + m}{m} a = \frac{2(2M + m)s}{mt^2} \text{ cm. per sec. per sec.}$$

The quantities on the right-hand can all be determined and hence a value can be found for g.

In an experiment the following values were obtained. It was found that the masses traversed 210 cm. in 10 seconds.

Hence $$a = \frac{2 \times 210}{10^2} = 4 \cdot 2 \text{ cm. per sec. per sec.}$$

The masses were $M = 1030$ grammes, $m = 9$ grammes.

Hence $$g = \frac{2 \times 1030 + 9}{9} \times 4 \cdot 2$$

$$= 966 \text{ cm. per sec. per sec.}$$

Sources of error.

The main sources of error are two. (1) The pulley has mass which though small may be appreciable; thus the whole rate of change of momentum is not $(2M+m)\,a$, but this quantity together with something depending on the pulley. Now the outer edge of the pulley moves with the same velocity as the descending masses, if we call this v we may represent the momentum of the pulley by $M'v$, where M' is not the mass of the pulley but is a mass which if concentrated in the rim of the pulley and moving with its actual velocity would have momentum equal to that of the pulley. M' is clearly less than the mass of the pulley, for some parts of the pulley are moving with a velocity less than v. Its exact value will depend on the distribution of mass in the various parts of the pulley. If, as is usual, nearly the whole mass is in the rim, M' will not be much less than the mass of the pulley. The rate at which the momentum of the pulley is changing is $M'a$. Hence the left-hand side of the equation of motion would be

$$(2M + m + M')\,a.$$

The numerator of the fraction giving the value of g should be increased by M' and the value of g should be larger.

(2) Again, owing to the friction the force acting is not mg but something less; the acceleration observed is less than it would be if there were no friction, the value of g found is in consequence too small. For a method of correcting for this, see Glazebrook and Shaw, *Practical Physics*, § 21.

EXAMPLES.

LAWS OF MOTION.

1. Calculate the momenta of a mass of 1 kilogramme when moving with the following velocities:

(i) 1 metre per second, (ii) 75 cm. per hour,
(iii) 1 mile per minute, (iv) after falling 500 metres.

2. Compare the momenta of a bullet whose mass is $\frac{1}{2}$ an oz. moving with a speed of 1000 feet per second and of a mass of 60 kilogrammes whose speed is 1 kilometre per minute.

3. A mass of 60 kilogrammes acquires in moving for one minute a speed of 1 kilometre per minute. Find the acceleration and the impressed force, stating clearly the units in which each is measured.

4. Compare the impressed forces (supposed uniform) on the two bodies mentioned in question 2, assuming the bullet to have gained its speed in $\frac{1}{8}$ of a second and the large mass in 5 minutes.

5. Find the momentum of the Earth, taking its mass as 5×10^{27} grammes and assuming it to describe a circle of radius 92000000 miles in a year.

6. A bullet whose mass is 150 grammes, moving with a speed of 500 metres per second, strikes and remains imbedded in a lump of soft wood whose mass is 25 kilogrammes suspended by a string 1 metre long. Through what height does the wood swing?

7. A cricket ball moving with a speed of 30 feet per second is hit to square leg, and after the blow moves with double the speed. Find the impulse, and assuming contact with the bat to last for $\frac{1}{50}$ of a second, find the average force.

8. A spiral spring in expanding through 25 cm. can exert an average force equal to the weight of 1 kilogramme. Find the velocity it will produce in a mass of 10 grammes with which it is in contact through the whole distance.

9. A particle starting from rest is acted on by a force equal to the weight of ten pounds. After twelve seconds the velocity is five yards a second, find the mass and the weight of the particle.

10. A constant force acts on a particle in the direction of its motion, state the relation connecting the increase of momentum and the time during which the force has acted.

11. A mass of 10 lb. moving with a velocity of 25 feet per second is stopped by a uniform force in a distance of 50 feet: find the force.

12. A force equal to the weight of 10 grammes acts on a mass of 27 grammes for 1 second: if the value of g be 982, find the velocity of the mass and the space it has travelled over. At the end of the first second the force ceases to act; how much further will the body move in the next minute?

13. A force equal to the weight of 10 lb. acts upon a mass of 8 lb., the mass moves vertically upwards with uniform acceleration. What will be its velocity at the end of the third second starting from rest?

14. A force equal to the weight of 12 lb. acts on a heavy body which moves vertically upwards with uniform acceleration. If the body passes over 18 feet in three seconds starting from rest, find the mass of the body.

15. A force of a pound weight acting upon a certain body for a minute generates in it a velocity of 60 miles an hour: find the mass of the body.

16. How far will a railway carriage starting with a velocity of 60 miles per hour run on level rails if the resistance be ·002 of its weight?

17. A train going at 30 miles per hour pulls up in 200 yds.; what is the direction and magnitude of its acceleration?

If every wheel skids, find the resistance in terms of the weight, supposing the line to be level.

18. Describe some accurate method of determining the value of the acceleration due to gravity. How would you arrange an experiment to make a body fall with an acceleration each second of one foot per second?

19. What is the relation between the force producing motion in a given mass and the motion produced? How would you verify this relation?

20. A mass has its velocity changed (i) from rest to 10 feet per second, (ii) from 10 feet to 20 feet per second. Compare the magnitudes of the forces required, the time occupied in the change being in each case 5 seconds.

21. Describe an experiment to shew that the rate of change of momentum of a moving body is proportional to the force producing motion. What force is needed to reverse the motion of a mass of 1 lb. moving with a speed of 100 feet per second, supposing the whole change to take place in 1 second?

22. A weight of 100 lb. is placed on a smooth horizontal table; what force acting horizontally for three seconds will generate in it a velocity of 64 feet per second?

23. Shew that, if the force on a body be taken to be numerically expressed by the product of the numbers expressing its mass and acceleration respectively, the unit of force is dependant on the units of mass, length, and time. Indicate the unit of force thus defined when the gramme, centimetre, and second are the units of mass, length, and time respectively.

24. Describe Atwood's Machine. How would you prove by means of it that the space described from rest by a body acted on by a uniform force varies as the square of the time?

25. Two masses of 8 and 10 ounces are connected by a string passing over a pulley; find the tension of the string when they are in motion, and the space described in 4 seconds.

26. A spiral spring, which for every millimetre of extension requires a force of 20 grammes weight, is hung up by one end and a mass of 50 grammes is attached to the other end by a long string. If the mass is raised and allowed to fall so that it travels a distance of 30 centimetres before the string becomes tight, find what extension of the spring will be produced.

27. Two masses of 8 lb. and 10 lb. are supported by a fine string over a smooth pulley. After falling through 5 feet the 10 lb. weight strikes the floor, find the impulse on the floor.

28. Masses of 1 and 3 lb. hang from the two ends of a fine string suspended over a smooth pulley. At what rate will they be moving at the end of 1 sec. after they are set free?

29. Weights of five and ten grammes are connected by a string which passes over a pulley: if the weights are allowed to fall, find their velocity when the heavier weight has descended through a metre.

30. Find the tension of a rope which draws a carriage weighing 1 ton up an incline of 30° with a velocity which increases by 1 foot per second per second.

31. When one weight lifts another by means of a string passing over a fixed pulley without friction, find the tension of the string and the velocity produced in one second.

If the first weight be 4 lb. and the second 2 lb. what is the tension of the string in lb.-wt.? and what the velocity produced per second in feet per second?

32. Masses of 3 lb. and 3 lb. 1 oz. respectively are attached to the ends of the string of an Atwood's machine and move from rest during 4 seconds; the 1 oz. is then removed and it is found that the masses move over 8 feet in the next 5 seconds. Find the numerical value of gravity which results from these experiments.

33. Each of the masses attached to the ends of the string of an Atwood's machine are 150 grammes; if the inertia of the pulley be taken as equivalent to an additional mass of 25 grammes, find the acceleration when a mass of 5 grammes is added to one side. The value of g may be taken as 980 cm. per sec. per sec.

34. Two weights each of 5 lb. are tied to the two ends of a long cord which hangs over a pulley in a vertical plane. An additional weight of 2 lb. is suddenly placed on one end. Find the acceleration of the weights, and also the velocity and amount of displacement after 3 seconds, neglecting friction and inertia of the pulley and the stiffness of the cord.

35. How may Atwood's machine be used to find the acceleration due to gravity? The masses on either side are 250 grammes; if the inertia of the pulley be taken as equivalent to an additional mass of 50 grammes, find the acceleration when a mass of 5 grammes is added to one side. The value of g may be taken as 980 cm. per sec. per sec.

36. A heavy particle slides down a smooth inclined plane, starting from rest at the top, the height of the plane being h and the length l feet. Find the acceleration of the particle, and the time it will take to reach the bottom.

With what velocity must a particle be projected down a plane 12 feet in height and inclined to the horizon at an angle of 30°, so as to reach the bottom in one second?

37. A heavy particle is projected up a smooth inclined plane whose height is h, and length l, with such a velocity that it just reaches the top of the plane. Find the acceleration of the particle and the time that it takes to mount.

With what velocity must a particle be projected up a plane 10 feet in height and inclined to the horizon at an angle of 30°, so as to reach the top in one second?

38. The line AB is vertical, and ACB is a right angle. Shew that the time of sliding down either AC or CB, supposed smooth, is equal to the time of falling down AB.

39. Let PQ be a chord of a vertical circle whose highest point is A and centre O. Then if the time of descent down this chord be half of that down the vertical diameter, shew that

$$\tan \tfrac{1}{2} AOP : \tan \tfrac{1}{2} AOQ :: 3 : 5.$$

40. Distinguish between a poundal and the weight of one pound.

What is the experimental evidence for the statement that the weight of a body is proportional to its mass?

41. Distinguish between mass and weight.

If the weight of a certain mass be represented by 15 at a place where a body falls through 64 feet in 2 seconds, what will be the weight of the same mass at a place where a body falls through 176 feet in 3 seconds?

42. The time of falling down a smooth inclined plane is twice that down the vertical height of the plane. Find the ratio of the length of the plane to its height.

43. If the wheel in an Atwood's machine is so stiff that a weight P on one side will just support nP on the other, find the acceleration when the weights are P and $n'P$ $(n' > n)$.

44. Find the change of momentum of a body of mass m which is initially moving with velocity u, and then has its velocity deflected through an angle α, but without change of magnitude.

Two particles A and B of the same mass m are connected by a light string and rest with the string just tight on a horizontal table whose coefficient of friction is μ. A uniform force P (greater than $2\mu mg$) begins to act on the particle A in the direction BA. Find the acceleration of either particle and the tension of the string at any time.

45. A mass of 10 grammes resting on a smooth table is connected by a string passing over a pulley with an equal mass which hangs vertically. The first mass is 1 metre from the edge; find the velocity of the system at the moment at which it is dragged over.

46. A mass of 200 grammes can just be dragged up an inclined plane by a mass of 100 grammes which is connected to it by a string passing over a pulley and hangs vertically from the top of the plane. Find the inclination of the plane; find also the velocity produced after the system has moved through 1 metre, if a mass of 5 grammes be placed on the smaller mass.

47. A mass of 1 lb. slides down an inclined plane whose height is half its length and draws a mass of 5 lb. along a smooth horizontal table level with the top of the plane. Find the acceleration; find also the momentum of the whole after the masses have moved 1 yard.

48. Two masses of 5 lb. and 10 lb. respectively are placed on two inclined planes of the same height and the angle 30°; the masses are connected by a fine string passing over a smooth pulley at the top of the two planes. Find the acceleration, the tension of the string, and the distance traversed in 5 seconds.

49. An engine draws a train whose mass is 75 tons up a slope of 1 in 50. If the resistance of the rails be $\frac{1}{80}$ of the weight, find the force exerted by the engine in order that the speed may be constant. The speed at the bottom of the slope is 25 miles an hour: if the engine stops working how far will the train run?

50. A mass of 10 kilogrammes slides down a rough plane which rises 1 in 10: find the resistance if the speed remains constant.

51. What is meant by the equation $W=Mg$?

A lift is rising with an acceleration of 8 feet per second per second; what pressure would a man scaling 16 stone exert on the floor of the lift?

52. A particle, starting from rest, slides down a smooth plane inclined at an angle of 45° to the horizon in 6 seconds; find the length of the inclined plane.

53. Two equal masses are at rest side by side. One moves from rest under a constant force F while at the same instant the other receives an impulse I. Shew that they will again be side by side after a time $\frac{2I}{F}$.

54. A particle whose mass is 10 lb. is moving with a velocity of 80 feet per second. If it is brought to rest in 100 feet by applying a constant resistance, find the magnitude of the resistance.

55. A particle whose mass is 12 lb. is moving with a velocity of 25 feet per second. If a constant resistance equal to a weight of 15 oz. is applied to stop it, find how far it will travel before it comes to rest.

56. A weight of 7 lb. is placed on a smooth horizontal board and connected by strings passing over smooth pulleys at each end of the board with weights of 5 lb. and 4 lb. respectively. Find the acceleration of the weights and the tensions of the strings.

57. If two masses m, m' are connected by a string, whose mass can be neglected, passing over a smooth fixed pulley, find the tension of the string.

Two bodies, of mass 2 lb. and 30 lb. respectively, lie on a smooth horizontal table whose height above the floor is 27 inches. The bodies are connected by an inextensible string, whose length is not less than 27 inches, and, when the string is taut, the smaller mass is dropped through a hole in the table. Find when it reaches the ground.

CHAPTER VIII.

THE THIRD LAW OF MOTION. ENERGY.

99. Action and Reaction. In the Third Law of Motion Newton stated that *To every action there is always an equal and opposite reaction, or the mutual actions of two bodies are always equal and opposite.*

The Law is based on observation and experiment, in considering it however we are at once met by the question What is meant by **Action**? We can learn from Newton's own illustrations what he understood by the term.

"If a man presses a stone with his finger," he says, "his finger also is pressed by the stone. If a horse draws a stone by means of a rope the horse is drawn equally towards the stone, for the rope stretched between the two will urge the horse towards the stone and the stone towards the horse; and this will impede the progress of the one as much as it helps that of the other."

"If a body impinging on a second body changes the motion of that body in any manner by the force it exerts, it will itself undergo the same change in its own motion in the opposite direction through the force exerted by the second body (because of the equality of the mutual pressure)."

"It is the amount of *momentum*, not of *velocity*, interchanged in these actions which is equal, at least in bodies which are otherwise unimpeded. For the changes of velocity which take place also in opposite directions are reciprocally proportional to the [masses of the] bodies, since the change of

momentum is the same for the two. The law is also true for the case of attractive forces."

100. Illustrations of the Third Law. These illustrations of Newton's make it clear that the action and reaction contemplated by him was the *Interchange* of *Momentum* between two bodies. We shall see later that, in a scholium attached to the law, he interprets the terms in another sense in which they are equally true.

If then we are to mean by Action *Gain of Momentum*, the experiments on impact described in Sections 51—61 afford a verification of the law.

Moreover, when two bodies are unimpeded in their action on each other, not only is the whole gain of momentum of the one equal to the whole loss of the other, but, during the time of transfer, the rate at which the one gains momentum is equal to the rate at which the other loses it. The impressed force on the one is equal and opposite to that on the other. The Moon has an acceleration towards the Earth and the Earth towards the Moon, astronomical observations shew that these accelerations are inversely proportional to the masses of the Moon and the Earth respectively. For Moon and Earth the equation $ma = m'a'$ holds, m and m' being the respective masses, a and a' the respective accelerations. Each gains momentum at the same rate. An experiment, also due to Newton, may illustrate this point. Float a magnet on a block of wood in a large[1] vessel of water; float a piece of soft iron on a second block of wood. Place the iron at a little distance from the magnet and in such a position that the magnet may point directly towards it. Hold the two at rest until the water is quite still, then release them simultaneously. The magnet will draw the iron to itself, the two blocks of wood will impinge and then come to rest[2] on the water.

The two blocks gain momentum in opposite directions when free to move, this gain however is the same for each, the force

[1] Large because otherwise capillary action at the sides of the vessel may affect the result.

[2] It is desirable in performing this experiment to arrange that the impact may be direct, otherwise the blocks of wood may be made to spin round in the water.

on the one is equal and opposite to that on the other; after impact the system remains without momentum.

The following also affords us an illustration of this law. When a gun is fired it kicks, unless held close to the shoulder it bruises it. If the gun be held firm, it and the man constitute, so far as the action is concerned, but one body, the mass of which is much greater than that of the bullet; the velocity acquired by this body will be small compared with that of the bullet, but the momentum of the bullet and of the gun and man combined will be equal and opposite. We can shew by the following experiment that this is so.

The gun is attached with its barrel in a horizontal position to a massive block of wood which can swing about a horizontal axis above the gun and at right angles to the barrel; the whole constitutes a pendulum; as with the simple pendulum if the gun be displaced from its equilibrium position its velocity can be calculated by observing the arc through which it swings, the velocity of the bullet as it leaves the barrel can also be found. Suppose the gun loaded and fired when at rest at the lowest point of its swing; observe the velocities of the gun and pendulum and of the bullet and calculate the momentum of each: it will be found that they are equal.

A simpler experiment of the same character is as follows.

EXPERIMENT 22. *To illustrate by experiment the third law of motion.*

Suspend two balls, Fig. 64, of different masses, so that their centres may be at the same level and equally distant from their points of support; compress a fairly strong spiral spring and tie it with thread so that the spires are as close together as possible. Place it between the balls; release the spring by cutting or burning the thread, the balls will move in opposite directions; determine their velocities by observing the distances through which each ball swings, it will be found that the velocities are inversely proportional to the masses of the balls. Each ball acquires the same amount of momentum.

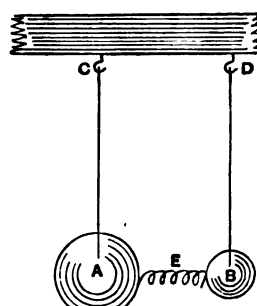

Fig. 64.

Hicks' ballistic balance already described may be used for this experiment, with this apparatus the velocities are easily measured.

One case mentioned by Newton needs a little further consideration; when a horse drags a load or tows a boat how is it that the horse is pulled back as much as the load is pulled forward? The horse acquires momentum by the action between his feet and the road. If he is moving with uniform speed and we neglect the relative motion of the various parts of his body the whole of the momentum so acquired is transferred by the rope to the boat. The passage of the boat through the water is resisted and the boat loses momentum to the water, this loss is equal to the gain it acquired from the horse; thus the momentum gained by the horse appears in the water wave which follows the boat. Now the Earth has lost momentum—estimated in the direction in which the horse is moving—equal to that gained by the horse: by the actions just described the water forming part of the Earth has gained the same amount in the direction of the horse's motion, thus on the whole there is neither loss nor gain. If the boat is gaining speed the horse acquires from the ground rather more momentum than is transmitted by the rope to the boat.

***101. Attraction and the Third Law.** This equality of action and reaction holds also in the case of the **Attractions** between the various portions of the same body.

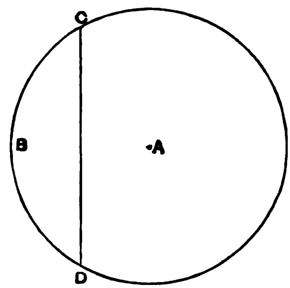

Fig. 65.

For consider a body like the Earth, imagine it free from external action, and suppose it divided into two parts A, B, Fig. 65, by a plane CD; if it be possible let A gain momentum from B faster than B gains it from A, then the whole system is continually acquiring momentum in the direction from B to A and will move off with continually increasing velocity into space in the direction BA; this is contrary to the first law for the body is free from external action.

Example. As an example of the third law consider the case of a shell exploding in the air. The fragments are projected in various directions but the total gain of momentum of the various parts is zero; the momentum gained in one direction by some parts is equal to that gained in the opposite direction by others. Thus if the shell burst into two equal pieces, if its original velocity be 100 feet per second, and the velocity after fracture of one part in the same direction be 150 feet per second we have, calling m the mass of the whole shell and v the velocity of the other part,

$$m \times 100 = \frac{m}{2} \times 150 + \frac{m}{2} v.$$

Hence $v=50$, the one half gains $25m$ units of momentum, the other loses the same amount and continues to move forward with a velocity of $100-50$ or 50 feet per second.

We see by these and similar examples how to interpret the third law of motion if we give to action the meaning of change or rate of change of momentum.

102. Conservation of Momentum. In all these cases then **Action is Transference of Momentum.** Whenever a transference of momentum takes place between two bodies the loss of the one body is equal to the gain of the other.

From this point of view the third law expresses the **Conservation of Momentum.**

We can observe the facts that the one body gains momentum while the other loses it; we call the rate at which this transference takes place **Force,** *and when the transference is going on we say that* **Force acts** *between the two bodies, its action on the two being equal and opposite.*

103. Action.—Energy. But the third law as Newton pointed out contains more than this.

We shall find that, when bodies act on each other, another quantity (**Energy**) besides Momentum is transferred, and that in this case too there is neither gain nor loss on the whole transaction.

In the Scholium to the laws of motion, Newton calls attention to the importance of the quantity obtained by multiplying force and the displacement of its point of application. He writes after giving some examples—"By these I wished to shew how wide-spread and how certain is the third law of motion. For if the action of an agent be measured by the product of its force and displacement[1], and similarly the reaction of the resisting body be measured by the sum of the products of the resistances and their several displacements, then whether the resistances arise from friction, cohesion, weight or acceleration, in all cases action and reaction

[1] The word actually employed is velocity, but the velocities concerned are those which occur during the same moment, they are measured therefore by the displacements.

in every kind of machine will be equal." **Action** then in the third law may be measured by the *product of a force and the displacement of the point at which it is applied*; this displacement however as Newton points out is to be estimated in the direction of the force.

In other words, when momentum is being transferred to a body at a given point, the product of the rate at which the body is gaining momentum and the displacement of the point at which the transference is taking place may measure the Action which is considered in some aspects of the third law.

The meaning and importance of this statement will however be better appreciated after some preliminary explanations and definitions.

104. Work and Energy. When momentum is being transferred to a body and the point at which the transference is taking place is in motion, we say in general that **Work** is being done.

Now let F be the amount of momentum transferred per second—the force; let A, Fig. 66, be the point at which the transference is taking place—the point of action of the force. Let AB be the direction in which the momentum is being transferred—the direction of the force. Let A be displaced to A' and draw $A'A_1$ perpendicular to AB to meet it in A_1.

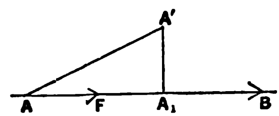

Fig. 66.

Then the work done is measured by the product $F \times AA_1$.

The actual displacement of the point A is AA', this can be resolved into AA_1 in the direction of the force and A_1A' at right angles to the direction of the force, when this is done AA_1, the component of the displacement in the direction of the force, is spoken of as the displacement in the direction of the force and the work done is the product of the force into the displacement in the direction of the force.

105. Measurement of Work.

DEFINITION. *When* **Momentum** *is being transferred to a body at a point which is in motion the* **Work** *done is measured by the product of the rate at which the body is gaining momentum at that point multiplied by the component of the displacement in the direction of the momentum.*

Or, in other words. The product of a Force into the component in its own direction of the displacement of its point of application measures the Work done.

Thus let F be the force, let the point of application of the force be displaced a distance s in the direction of the force; the work done U is given by the equation

$$U = Fs.$$

When the actual displacement s is not in the direction of the force we resolve it, as in Figure 66 above, into s_1 in that direction and s_2 at right angles to the direction, in this case the work done is given by Fs_1. If the angle A_1AA' be called θ, then since $A'A_1A$ is a right angle we have from Figure 66,

$$A_1A = AA' \cos\theta.$$

Hence $$s_1 = s\cos\theta.$$

Therefore $$U = Fs\cos\theta.$$

Now in this expression $F\cos\theta$ is the component of the force in the direction of displacement. Thus we see that the work is found by multiplying together either the force and the component of the displacement in the direction of the force or the displacement and the component of the force in the direction of the displacement.

Hence

Work = Force × component of displacement in the direction of the force.

= Displacement × component of the force in the direction of the displacement

$$= Fs\cos\theta.$$

If, as in Fig. 67, the displacement AA' be at right angles to the force then the displacement in the direction of the force is zero, thus the work done is zero.

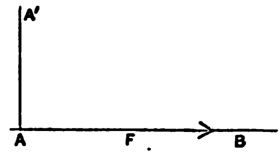

Fig. 67.

Hence, when the motion of the point of application is at right angles to the direction of the force, no work is done.

Again, the component of the displacement with which we are concerned may be as AA_1 in Fig. 68 in the same direction as that in which the force acts or it may be as AA_1 in Fig. 69 in the opposite direction to that of the force. In the first case, Fig. 68, work is done *on* the body *by* the force, in the second, Fig. 69, work is done *by* the body *against* the force.

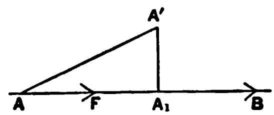

Fig. 68.

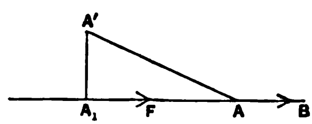

Fig. 69.

Thus when a body is gaining momentum at any point and when the direction of motion of the point is the same as that of the gain of momentum work is done *on* the body; when the direction of motion of the point is opposite to that of the gain of momentum work is done *by* the body; when the direction of motion is at right angles to that of the momentum no work is done.

Hence, if a body, acted on by gravity only, move in a horizontal direction with uniform speed no work is done, if the body be raised work is done on the body against its weight by the agent raising it, if the body be allowed to fall from a height work is done by its weight.

When a body of mass m is raised a height h the upward impressed force is mg and the work done is mgh.

In making this statement it is supposed that the body is raised very slowly. The upward force is really greater than mg, otherwise the body would not move, but a force which is just in excess of mg will raise it; if the excess be extremely small the motion will be exceedingly slow, and

the work done will differ from *mgh* by a very small quantity which we may neglect. If the body is drawn up with a run more work will be necessary. See Section 112.

It should be noticed that two factors are necessary to measure work. It depends on the product of the force or rate of transference of momentum and of the displacement. If we know the amount of work done, we cannot calculate the force unless we also know the displacement: a small force working through a large distance may do as much work as a large force working through a small distance, the work is the product of the two.

Again, the work done does *not* depend on the time in which it is done. An engine which can raise a ton a foot in the course of a year will then have done as much work as one which raises a ton the same distance in a second; the amount of work done depends solely on the product of the force and the displacement, and is quite independent of the time taken to do it.

106. Unit of Work. Consider now a body which is gaining F units of momentum per second; if the body be displaced a distance s in the direction of this momentum, then the work done is U, where

$$U = F \,.\, s.$$

If then the force or rate of gain of momentum be unity and the displacement also be unity the work done is unity.

Thus the **Unit of Work** is done when a particle on which unit force is acting is displaced unit distance in the direction of the force.

The value then of the unit of work depends on the unit of force and on the unit of length.

107. The C. G. S. Unit Work. On the C. G. S. system the unit force is a Dyne and the unit distance a Centimetre; the C. G. S. unit of work then is done when a particle on which 1 Dyne is acting is displaced 1 Centimetre in the direction of the force, this unit of work is called an **Erg.**

DEFINITION. *One* **Erg** *is the work done when the point of application of a force of* 1 *dyne is moved* 1 *centimetre in the direction of the force.*

The Erg is a very small unit of work for a dyne is a very small unit of force, being rather less than the weight of 1 cubic millimetre of water. Hence, the erg is rather less than the work done in raising a cubic millimetre of water 1 centimetre. For this reason ten million ergs are taken as the practical C. G. S. unit of work and are called a **Joule.**

Thus we have

$$1 \text{ Joule} = 10{,}000{,}000 = 10^7 \text{ Ergs.}$$

108. The F. P. S. Unit Work. On the F. P. S. system the unit force is one poundal. Unit work then is done on this system when the point of application of a force of 1 poundal is moved through 1 foot. This unit is called the Foot-poundal.

DEFINITION. *One* **Foot-poundal** *is the work done when the point of application of a force of* 1 *poundal is moved* 1 *foot in the direction of the force.*

109. Gravitational units of work. Another unit of force used is the weight of a body whose mass is 1 gramme. When the point of application of such a force is moved 1 centimetre, **1 centimetre-gramme unit of work** is done. Since the weight of a gramme contains g dynes we see that 1 centimetre-gramme unit of work contains g ergs.

Moreover since g depends on locality the centimetre-gramme unit of work is different at different points of the Earth, the erg is the same everywhere.

DEFINITION. *One* **centimetre-gramme unit of work** *is done when the point of application of a force equal to the weight of* 1 *gramme is moved* 1 *centimetre in the direction of the force.*

Hence, 1 centimetre-gramme unit of work

$$= g \text{ ergs} = 981 \text{ ergs},$$

if we take g as 981 centimetres per second per second.

Again, the weight of a mass of 1 pound is taken sometimes as the unit of force. When this is done the corresponding unit of work is the foot-pound.

DEFINITION. *One* **foot-pound unit** *is the work done when the point of application of a force equal to the weight of* 1 *pound is moved* 1 *foot in the direction of the force.*

Since the weight of 1 pound contains g poundals we see that one foot-pound is equal to g foot-poundals, and since in feet per second per second g is 32·2, we see that

$$1 \text{ foot-pound} = g \text{ foot-poundals}$$
$$= 32{\cdot}2 \text{ foot-poundals}.$$

A foot-pound of work is done when a mass of 1 pound is raised 1 foot.

The work done in raising a kilogramme through one metre is 1000×100 or 100,000 centimetre-gramme units. Since each of these contains 981 ergs the work is 98,100,000 ergs.

110. Rate of Working. The rate at which work is done is called **Power.**

DEFINITION. **Power** *is measured, when uniform, by the work done per second, when variable by the ratio of the work done in an interval of time to that interval when it is sufficiently small*[1].

Thus the powers of the two engines mentioned in Section 105 are very different. The second has a power of 1 foot-ton or 2240 foot-pounds per second, while the first, since there are 31536000 seconds in the year, has a power of 2240/31536000 foot-pounds per second.

A **Horse-Power** is the name given to a unit of power in common use.

DEFINITION. *When* 550 *foot-pounds of work are being done per second the rate of working is* 1 **Horse-Power.**

[1] See Section 22.

Thus the second engine is one of rather over 4 horse-power (really 224/55 horse-power).

The rate of working in common use on the C.G.S. system is **The Watt.**

DEFINITION. *When* 1 *Joule* (10^7 *ergs of work*) *is being done per second the* **Rate of Working** is 1 **Watt.**

Thus 1 Watt is 10^7 ergs done per second. We can shew[1] that a Joule is about ·737 of a foot-pound, so that a Watt is ·737 foot-pound per second. *Thus a Horse-power is* 746 *Watts.*

111. Measurement of Power. Since work is measured by force multiplied by displacement, if the force be constant the rate of working is measured by force multiplied by rate of displacement. Now rate of displacement is velocity.

Hence **Power** is measured by *the product of a force into the velocity of its point of application measured in the direction of the force.* In other words, the rate at which work is being done on a particle is the product of its rate of gain of momentum and the component of its velocity measured in the direction in which it is gaining momentum. Thus if F be the force impressed on a particle and v_1 the component of its velocity in the direction of F. Then Rate of Working $= Fv_1 = Fv \cos\theta$, if v be the velocity and θ the angle between the directions of v and F.

112. Expressions for Work and Power. The expressions which have been found for Work and Power may be put into various forms. Thus

PROPOSITION 27. *To shew that if a body of mass* m *acquire a velocity* v *in moving with constant acceleration in a straight line from rest through a space* s *the work done is* $\frac{1}{2}mv^2$.

Let F be the impressed force, a the acceleration, U the work. Then we have $F = ma$ and $v^2 = 2as$.

Hence $U = Fs = mas = \frac{1}{2}mv^2$.

Again, if the initial velocity be u and not zero, we have $v^2 - u^2 = 2as$.

Hence $U = Fs = mas = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$.

[1] See Section 112, Example 4.

The units in which this result is given will depend on those in which m and s are measured, for instance, on the C.G.S. system m is in grammes and s in centimetres; since we apply the equation $F=ma$ the force is in dynes, hence the work Fs is in centimetre-dynes, or ergs.

Thus the work required to give to a mass of m grammes a velocity of v centimetres per second is $\frac{1}{2}mv^2$ *ergs*. If the mass be in pounds, the space in feet, then the work is in foot-poundals. If we wish to use gravitation measure we must remember that a dyne is $1/g$ of the weight of one gramme.

Hence $\frac{1}{2}mv^2$ ergs is $\frac{1}{2}mv^2/g$ centimetre-grammes of work and $\frac{1}{2}mv^2$ foot-poundals is $\frac{1}{2}mv^2/g$ foot-pounds where g is 981 cm. per sec. per sec. in C.G.S. units or 32·2 feet per sec. per sec. in F.P.S. units.

PROPOSITION 28. *To find expressions for the rate at which work is being done on a particle of mass* m, *moving in a straight line with constant acceleration* a.

Let v be the velocity of the particle at any moment, F the force, t the time from rest, and s the space traversed.

Then we have

$$\text{Power} = \text{Rate of Working} = Fv = mav$$

$$= \frac{mv^2}{t} = ma^2t = \frac{2mas}{t} = ma\sqrt{2as}.$$

Examples. (1). *Find in the various units the work done on a mass of* 1 *cwt. when lifted through* 100 *feet.*

Since $1 \text{ cwt.} = 112$ pounds,

$\text{work} = 112 \times 100$ foot-pounds,

$= 112 \times 100 \times 32{\cdot}2$ foot-poundals.

Also $1 \text{ lb.} = 453{\cdot}6$ grammes,

$1 \text{ foot} = 30{\cdot}48$ cms.

Thus $\text{work} = 112 \times 453{\cdot}6 \times 100 \times 30{\cdot}48$ centimetre-grammes

$= 112 \times 453{\cdot}6 \times 100 \times 30{\cdot}48 \times 981$ ergs,

and this reduces to about $1{\cdot}519 \times 10^{11}$ ergs or 15190 Joules.

(2). *This same mass is allowed to fall from a height of* 100 *feet. Calculate the work done by gravity* (*a*) *after it has fallen* 50 *feet,* (*b*) *when it has reached the ground, and determine in each case the rate at which work is being done.*

The rate at which the mass is gaining momentum or the force is $112 \times 32{\cdot}2$ poundals.

Thus the work which has been done in 50 feet is $112 \times 32{\cdot}2 \times 50$ foot-poundals, and in 100 feet it is twice as much, or $112 \times 32{\cdot}2 \times 100$ foot-

poundals. This last expression is the same as the work done in lifting the body to the height of 100 feet.

The rate of working is the product of the velocity and the rate at which momentum is being communicated. After falling 50 feet the velocity is $\sqrt{2 \times 32{\cdot}2 \times 50}$ feet per second, and the force is $112 \times 32{\cdot}2$ poundals, thus the rate of working is

$$112 \times 32{\cdot}2 \times \sqrt{2 \times 32{\cdot}2 \times 50} \text{ foot-poundals per second.}$$

This reduces to $2{\cdot}033 \times 10^5$ foot-poundals per second; if we wish to work in foot-pounds per second we have for the power $112 \times \sqrt{2 \times 32{\cdot}2 \times 50}$ or

6349 foot-pounds per second.

Dividing this by 550 we find for the horse-power 11·54.

Thus when a body of 1 cwt. in mass has fallen freely through 50 feet, work is being done on it at the rate of 11·54 horse-power.

When the body has fallen 100 feet we shall have to substitute 100 for 50 in the above formulæ,—the velocity will be $\sqrt{2 \times 32{\cdot}2 \times 100}$ feet per second, and we find for the rate of work 8964 foot-pounds per second or 16·17 horse-power.

(3). *Two bodies 1·5 kilos and 1 kilo in mass respectively, suspended by a fine string over a pulley are free to move. Find the work done 5 seconds after motion has commenced, and the rate at which it is then being done.*

The acceleration is given by

$$a = \frac{1{\cdot}5 - 1}{1{\cdot}5 + 1} g.$$

Thus $a = g/5$ cm. per sec. per sec.

The rate at which the system gains momentum in the downward direction is $(1500 - 1000)\,g$ dynes, and this reduces to $500g$ dynes.

In 5 seconds the velocity (at) is $5 \times g/5$, or g cm. per second, and the space traversed $\frac{1}{2}at^2$ or $25g/10$ cm.

Thus the work done is

$$\frac{500g \times 25g}{10} \text{ or } 50 \times 25 \times g^2 \text{ ergs.}$$

This reduces to $120{\cdot}3 \times 10^7$ ergs or 120·3 Joules.

The rate of working being the product of the force and the velocity is

$$500g \times g \text{ ergs per second,}$$

or $$48{\cdot}12 \times 10^7 \text{ ergs per second.}$$

This is 48·12 watts.

(4). *Find the value of an erg in foot-poundals and of a horse-power in watts.*

$$1 \text{ erg} = \frac{1}{981} \text{ centimetre-gramme unit,}$$

$$1 \text{ centimetre} = \cdot 03281 \text{ feet,}$$

$$1 \text{ gramme} = \cdot 002205 \text{ lbs.}$$

$$1 \text{ erg} = \frac{\cdot 03281 \times \cdot 002205}{981} \text{ foot-pounds.}$$

This reduces to ·00000007374 foot-pound.

Hence 1 Joule is ·7374 foot-pound.

Thus
$$1 \text{ foot-pound} = \frac{1}{\cdot 7374} \text{ Joules}$$
$$= 1 \cdot 356 \text{ Joules.}$$

$$1 \text{ horse-power} = 550 \times 1 \cdot 356 \text{ Joules per second}$$
$$= 745 \cdot 8 \text{ watts}$$
$$= \text{approximately } \tfrac{3}{4} \text{ of a kilowatt.}$$

113. Measurement of Work. In the above examples we have shewn how to calculate the work done in various cases in which a particle—or a body which we may treat as a particle—is displaced in the line of action of the force; we will now consider some cases in which the displacement is oblique to the force. Suppose then that a body of mass m is moved from a point A to a point B, Fig. 70, where B is not vertically above A. Let the motion take place along the line AB and let us calculate the work done against gravity. The weight of the body is mg and its direction is vertical. If then we draw AC vertical and BC horizontal meeting in C, the displacement may be resolved into AC vertical in a direction *opposite* to that of the weight and BC horizontal in a direction at right angles to that of the weight. By the definition of work then the work done *against* the weight is $mg \times AC$ or mgh, if h is the vertical distance between A and B.

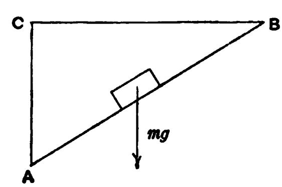

Fig. 70.

For the purpose of calculating the work, we may look upon the displacement as a vertical one AC $(=h)$ in which mgh units of work are done, and a horizontal one CB in which no work is done.

In raising a body from one point to another the work done against its weight depends only on the difference of level between the two points.

Moreover this result is independent of the path described by the body: if it be first moved vertically downwards its weight will do work but this will be cancelled by the work done against the weight in raising the body to its original position. If the path from A to B be a broken one as shewn in Fig. 71

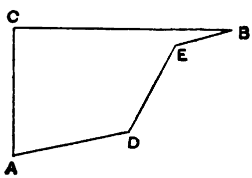

Fig. 71.

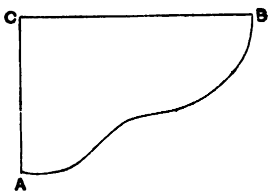

Fig. 72.

or a curved one as in Fig. 72, the work done is still the same, the vertical component of the whole displacement is in all cases AC or h and the work therefore is mgh.

Thus when the weight of the body is the only force considered, the work done in moving the body from one position to another against its weight depends only on the relative position of the two points—being proportional to their vertical distance apart—and not at all on the path of the particle between the points.

Now it can be shewn that a similar statement is true for a very large number of actually observed cases of motion. The work done on a body, which is gaining momentum by many of the processes which occur in nature, can be shewn to depend only on the initial and final positions of the body relative to such of its surroundings as influence its motion, and not at all on the path by which it has moved from one position to the other, or on the speed with which it has traversed this path.

There are some cases of motion for which this statement is not true. It is the fact however of its truth in many cases which gives to Work its great importance in Mechanics.

Action *then as used in the third law may mean the* **Work done on a body,** *reaction will be the work which a body can do in consequence of this, and the law states that these two, when all the forms of reaction are taken into consideration, are equal.*

In the *Statics* we shall have numerous examples of this principle and shall describe experiments arranged for its verification.

114. Motion of a body down a plane. Work. We have already found the value of the work done on a body by its weight when it is allowed to fall freely.

If m be the mass of the body, h the height through which it falls, and v the velocity it acquires, then U the work done is given by

$$U = mgh = \tfrac{1}{2}mv^2.$$

Let us now consider the work done by its weight on a body which is allowed to slide down a smooth inclined plane of height h. We know that the work done against its weight when the body is lifted from the bottom to the top of the plane is mgh.

PROPOSITION 29. *To find the work done on a body of mass* m *in sliding down a smooth inclined plane of height* h.

Let BA, Fig. 73, be the plane making an angle α with the horizon. Draw BC vertical and AC horizontal, let R be the force between the plane and the body; the weight of the body is mg.

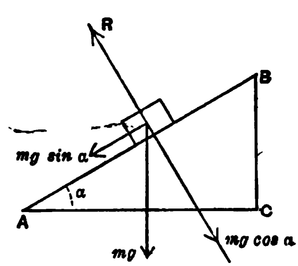

Fig. 73.

Then since the plane is smooth the direction of R is at right angles to it, the displacement of the body is along BA, at right angles to R, hence no work is done by the force R: the only force which does work is the weight mg, the displacement of

the body in its direction is h; hence the work done is mgh; thus an amount of work mgh is done on a body which slides down a smooth plane of height h.

Now let v be the velocity with which the body reaches the bottom, the acceleration parallel to the plane is $g \sin \alpha$: if l be the length of the plane we have a velocity v acquired in moving a distance l with acceleration $g \sin \alpha$.

Hence $\quad v^2 = 2gl \sin \alpha = 2gh, \quad$ for $h = l \sin \alpha$.

Thus $$mgh = \tfrac{1}{2}mv^2.$$

Hence the work done in sliding down the smooth plane which is equal to mgh is also given by $\frac{1}{2}mv^2$.

Moreover if the body be projected up the plane with velocity v it will just reach the top, and the work done against gravity will be mgh.

115. Work due to Gravity. We have just shewn that if a body be allowed to slide down a smooth inclined plane the velocity with which it reaches the bottom depends only on the height of the plane, and also that if the body starts up a second inclined plane with this same velocity it will rise to exactly the same height as that from which it fell.

Work is done by gravity on the body in sliding down; the body will rise again until this same amount of work is done against gravity.

Galileo discovered this relation between the velocity and height of fall and verified it by experiments which we will describe shortly.

We have deduced the above results on the supposition that the body slides down a *smooth flat* surface, a plane, so that its path is a straight line; it can be shewn that they are true if the surface be not flat but curved so that the path is a curve, not a straight line; for we may consider the curve as made up of a large number of very short straight lines inclined to each other at very small angles, the proposition is true for each of these lines; it is also true in the limit when they become a curve, though the proof of this would require some further consideration.

***116. Motion down a curve.** It is difficult to make observations on a particle *sliding* on a smooth curve;

no curve is perfectly smooth, and the corrections introduced by the friction are considerable.

We can however easily investigate motion in which the conditions are the same as on a curve. Thus, if a heavy body be suspended by a string and allowed to swing in a vertical plane it will move in a circle, the conditions of motion will be exactly the same as though it were sliding down the circular arc; the tension of the string acting at right angles to the path takes the place of the resistance of the curve. No work is done by this tension: if h be the vertical distance through which the body rises the work done is always mgh.

Consider now a body moving in such a circle; if by any means we can fix a point in the string, the body will continue to move in a circle but the radius of the circle will be less than before.

We can attain this result by allowing the motion to take place in front of a vertical wall or board; fix a nail or peg into the wall in such a way that the string may strike the nail, which will thus become the centre of the circle in

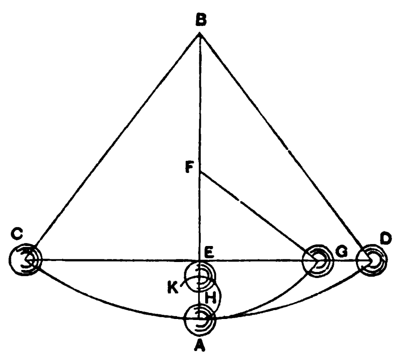

Fig. 74.

which the body will commence to move. Such an arrangement is shewn in Fig. 74; or again, if as in Fig. 75, we allow the string to unwrap itself off or to wrap itself on a curved surface such as FK, the path of the body will not be a

circle but will depend on the form of this curve. By properly adjusting this we can make the body to describe any path

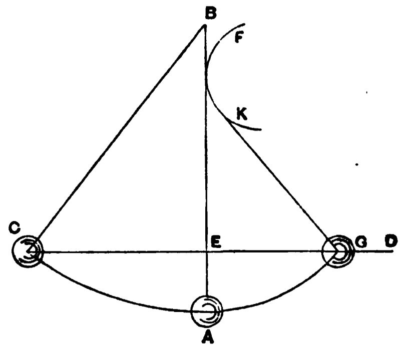

Fig. 75.

which we like and can thus investigate the motion of a body sliding down any smooth curve. In this way Galileo shewed that the *Height to which a body when moving under gravity on a smooth curve will rise depends only on the vertical height of its starting point above the lowest point of its path.*

EXPERIMENT 23. *To shew that a body moving under gravity in an arc of a vertical circle will ascend to the same height above the lowest point of the circle as that from which it started.*

Suspend a heavy body—an iron or lead sphere, some 6 or 8 cm. in diameter—by a long flexible cord such as a piece of waterproofed fishing-line about 2 metres long. Allow it to swing in front of a vertical wall or drawing-board about a point *B*, Fig. 74, and note the position *C* from which it starts. Observe the position *D* to which it rises at the end of its first swing and let *A* be the position it would occupy at rest. Join *CD* cutting *BA* in *E*, then it will be found that *CD* is horizontal[1]; the ball thus rises to the same height above *A* as the point from which it started.

[1] This statement is not quite strictly true, *D* will be a very little lower than *C*; the difference in height being due to the friction of the air for

Now drive a nail or peg into the wall as shewn at F a point in the vertical line AB, and again start the ball from C: when the string becomes vertical the portion BF is brought to rest, the ball proceeds to move in a circle AG with F as centre and rises to the position G before its motion stops. Observation will shew that G is in the horizontal line CD, the ball though now moving in a smaller circle than previously still attains the same height.

Repeat the experiment again, driving the nail in however at H a point between A and E, nearer to A than to E. Then the ball after the string has become vertical will describe a circle of radius HA about H. But this circle will not cut the horizontal line CD, the ball cannot rise to the same height as previously, it will be found that the ball completes the whole semicircle HK; its motion after passing through the point K will depend on the position of H and the radius of the circle. The ball may continue to describe the circle winding the string upon the nail, or the string may become slack for a time and the path of the ball alter.

Again, by reversing the direction of motion we may allow the ball to describe first the smaller circle with F as centre, then, when the string becomes vertical contact with F ceases, and the ball proceeds to move about A in the larger circle; it will in this case be found to rise to C, the same height as previously, and this will be the case for all positions of F which will permit the ball to start from some point in the horizontal line CD and describe an arc of a circle about F.

Thus in all these cases the velocity with which the ball starts up the circle AC must be the same. This velocity is acquired by sliding down the various circles corresponding to the different positions of F. Thus the velocity acquired by sliding down any of these circles from points in the horizontal line CD is the same.

Again, take a piece of wood[1], cut into the form of a curve

which no allowance has been made. It is possible to prove this and to make an allowance if required by experimenting with balls of the same size but of different material. With the apparatus as described, however, the correction will be very small.

[1] Instead of using wood the curve may be made out of a strip of sheet metal bent to the required form.

as shewn at FK, Fig. 75, and place it so that the string after passing the vertical comes into contact with the curve. The ball will no longer describe a circle but some curve, as shewn in Fig. 75, depending on the shape of the wood. It will be found however that it still comes to rest at the point G in which its path cuts the horizontal line through C the starting point, or that conversely, if it be started from G it will rise to C.

Thus the velocity acquired in sliding from rest under gravity a given vertical height down any curve is the same.

***117. Velocity on a curve.** The foregoing experiments enable us to find an expression for the velocity acquired in sliding down a curve; for we have seen that this is the same whatever be the form of the curve provided only that the height through which the body moves is constant. Now in Section 116 it has been shewn that in sliding down a smooth plane the velocity v acquired is given by the equation

$$v^2 = 2gh.$$

Thus in sliding from rest through a vertical height h down any curve a particle acquires a velocity v given by the equation

$$v^2 = 2gh.$$

The work done in this case is mgh and this is equal to $\frac{1}{2}mv^2$.

If the particle do not start from rest the corresponding equation may be found thus. Let u be the initial velocity and let it be acquired by sliding through a vertical height h'. Then the velocity v is acquired by sliding through a height $h + h'$.

Hence we have
$$v^2 = 2g(h + h'),$$
$$u^2 = 2gh'.$$

Thus subtracting
$$v^2 - u^2 = 2gh.$$

While for the work done we still have
$$U = mgh = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2.$$

This result corresponds exactly to those found in Section 112 for a body falling freely. It can be shewn that it is true for many cases of motion which are actually observed in nature. We may state then as a result of very wide application that when the velocity of a body changes from u to v work has been done on the body and

The work done is equal to $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$.

Hence, **Work** *must be done in order to increase the velocity of a body, and a body in having its velocity decreased can do work.*

Work *is also necessary in many cases to change the position of a body, while in changing its position the body can do work.*

118. Energy. Hence bodies, as we can observe them, have in some circumstances a capacity for doing work on other bodies; by their action momentum is communicated to those other bodies, which are thereby set in motion: work is done. This capacity for doing work is called **Energy.**

DEFINITION. *The* **Energy** *of a body is its capacity for doing work, and is measured by the work which the body can do in changing to some standard state as regards its position and velocity.*

It is sometimes more convenient to measure the energy of a body by the work which must be done on it to bring it to its actual state from some standard condition.

Thus a stone at the edge of a precipice has energy, a touch will send it over the edge to the ground below and in its fall it can do work; we can imagine it connected with another stone just lighter than itself by a fine string passing over a pulley; as it falls it can draw this other stone up. There are of course numberless other ways in which it could do work.

A body then has energy when raised above the Earth. For such a body it is usual to take as the standard state referred to in the definition that in which the body is at rest on the ground. *A body resting on the ground is said to have no energy.* When at a height h it has energy measured by the work done to lift it to that height; this, if the mass be m, is mgh.

DEFINITION. *The energy of the body which depends on its position and not on its velocity is called* **Potential Energy.**

Hence the potential energy of a mass m at a height h is mgh.

A moving body can do work in being stopped. It has energy in consequence of its motion. This form of energy is called **Kinetic Energy.**

DEFINITION. *The energy of a body which depends on its motion is called* **Kinetic Energy.**

The kinetic energy of a body is measured by the work which it can do in being brought to rest.

PROPOSITION 30. *To find the kinetic energy of a body which is moving with uniform velocity and is brought to rest by a uniform force.*

Let m be the mass of the body, v its velocity, F the force, s the distance the body will move before being stopped.

Then the work done in stopping the body is Fs.

Now we have

$$F = ma, \quad \text{and} \quad as = \tfrac{1}{2}v^2.$$

Thus

$$\text{Work} = Fs = mas = \tfrac{1}{2}mv^2.$$

Hence the kinetic energy of the body is $\frac{1}{2}mv^2$.

We may shew that this is a proper measure for the kinetic energy of the body in any case, and not merely when the retardation is constant.

Thus we have

$$\text{Kinetic Energy} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}mv \times v$$

$= \frac{1}{2}$ the product of the momentum and the velocity.

To prove this for a variable force we treat the force as uniform for very short intervals of time, but variable at the end of each interval. Let $F_1, F_2 \ldots$ be the values of the force and let $s_1, s_2 \ldots$ be the short distances traversed while the force has the values $F_1, F_2 \ldots$ etc. respectively. Then when s_1 etc. are very short the work actually done is

$$F_1 s_1 + F_2 s_2 + \ldots\ldots$$

Now let v_1, v_2 be the velocities at the beginnings of the spaces s_1, s_2 etc. Then during each of these spaces we may treat the retardation as uniform.

Hence
$$F_1 s_1 = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_2^2,$$
$$F_2 s_2 = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_3^2,$$
$$F_n s_n = \tfrac{1}{2}mv_n^2 - 0$$
if the body comes to rest at the end of the nth space s_n.

Thus adding these terms together $F_1 s_1 + F_2 s_2 + \ldots F_n s_n = \tfrac{1}{2}mv_1^2$.

Hence writing v for v_1 as the velocity of the particle we see that the work done in stopping the body is $\tfrac{1}{2}mv^2$. Thus $\tfrac{1}{2}mv^2$ is the kinetic energy.

119. Change of Form of Energy. The energy of a body may change its form from potential to kinetic and vice versa. Thus a stone at the top of a cliff is at rest but has potential energy mgh; just before it strikes the ground it is moving with velocity v and has kinetic energy $\tfrac{1}{2}mv^2$, but in this position it has no potential energy; we notice however that since the velocity v has been acquired in falling a distance h, we have $mgh = \tfrac{1}{2}mv^2$. Hence, *The kinetic energy at the bottom is equal to the potential energy at the top.* We shall find this result to be of the greatest importance: for the present let us consider some other transformations of energy. A bullet shot upwards from a gun starts with kinetic energy but with no potential energy; as it rises its kinetic energy decreases, for its velocity diminishes, but its potential energy increases, for its height becomes greater; when at the top of its flight it is instantaneously at rest; its kinetic energy is zero. The height to which it rises is found by dividing the energy with which it starts by its weight, for if v be the velocity of projection, K the kinetic energy at start, and h the height the bullet reaches, then
$$mgh = \tfrac{1}{2}mv^2 = K.$$
Hence
$$h = \frac{K}{mg} = \frac{\text{kinetic energy}}{\text{weight of bullet}};$$
as the bullet falls the potential energy becomes again transformed into kinetic energy.

A pendulum[1] bob when at the extremity of its swing has potential energy: if we take the equilibrium position as the standard one from which to measure, and if h be the height of the starting point above this position, then the potential energy is mgh. As the bob moves down to its equilibrium position its potential energy is diminished, its kinetic energy

[1] A simple pendulum is a ball at the end of a string.

increased until when at the lowest point the energy is all kinetic and is $\frac{1}{2}mv^2$; moreover the kinetic energy in this position is equal to the potential energy at starting, for $\frac{1}{2}mv^2 = mgh$. As the pendulum passes this equilibrium position and rises again, the transformation of energy takes place in the other direction, the kinetic energy becomes potential; we see moreover that it remains unchanged in amount since the pendulum rises to the height from which it started.

Many other examples of the transformation of energy might be given; we should find in all the same law, potential energy can be transformed into kinetic or kinetic into potential, but the gain of one is equal to the loss of the other. We will give a formal proof of this statement for one or two cases.

PROPOSITION 31. *To shew that the energy of a body falling freely remains unchanged during the fall.*

Let a body of mass m fall from a point A, Fig. 76, at a height h above the ground. Let v be its velocity when at the point P at a depth z below A, and E its total energy in this position. Then since PB is $h-z$ the height of the body is $h-z$.

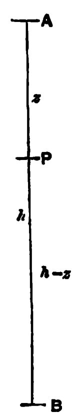

Fig. 76.

Hence its potential energy is $mg(h-z)$.

Its velocity is v; hence its kinetic energy is $\frac{1}{2}mv^2$.

Thus $$E = \tfrac{1}{2}mv^2 + mg(h-z).$$

But the velocity v is acquired by a fall through the distance z.

Therefore $$v^2 = 2gz,$$

and $$\tfrac{1}{2}mv^2 = mgz.$$

Hence $$E = mgz + mg(h-z)$$ $$= mgh.$$

Thus the energy in any position is mgh, which is equal to the potential energy at the start: the energy remains unchanged in amount during the motion.

We may put the proof slightly differently thus. In falling a depth z the body loses potential energy mgz, it gains kinetic energy $\frac{1}{2}mv^2$, and since $v^2 = 2gz$ these two are equal; thus the total energy does not change.

The same result follows if the body be projected down with a velocity u, instead of being dropped.

After it has dropped a depth z its total energy E is given as before by

$$E = \tfrac{1}{2}mv^2 + mg(h - z).$$

But $$v^2 = u^2 + 2gz.$$

Thus $$E = \tfrac{1}{2}mu^2 + mgh,$$

and this is the sum of the kinetic and potential energies at starting.

The same result is true when a body slides down a smooth curve; for in this case the same formulæ hold, h and z being the vertical distances between the various positions of the body.

***120. Mutual Energy.** We have thus arrived at the result that in a large number of cases of motion the energy of the moving body remains constant though it alters in form.

In the cases with which we have been dealing the energy depends on the position of the body relatively to the earth. We have determined the energy on the assumption that the Earth is at rest so far as the motion of the falling body is concerned and that the body falls to it; strictly of course this is not true, the Earth moves towards the body and the body towards the Earth, their accelerations being inversely as their masses; if we allow for this we find that it is the sum of the energies of the Earth and the body which remain constant. We ought not to speak of the potential energy of the body but of the mutual potential energy of the body and the Earth; in the fall some of this energy becomes transformed into the kinetic energies of the body and the Earth; the sum of these two is equal to the loss of mutual potential energy.

We are thus to look upon **Energy** as *a quantity which we can measure and which in such cases of motion as we have been considering remains unchanged during the motion.*

121. Forms of Energy. There are however other cases of motion in which energy apparently disappears. A falling stone just before reaching the ground has energy $\frac{1}{2}mv^2$, after striking the ground it is reduced to rest and has neither kinetic nor potential energy. Two masses which impinge

directly with equal momenta and adhere have kinetic energy before impact, they are reduced to rest and apparently lose this kinetic energy by the impact.

A body which is allowed to slide down a rough surface has potential energy at starting but it is soon brought to rest in a lower position; a railway train in motion has a large supply of kinetic energy; when the brakes are applied and the train stopped this kinetic energy has been dissipated.

Now it can be shewn that in all these cases the energy has merely changed its form. Heat has been produced and it is found that the heat produced is proportional to the energy which has disappeared. The experiments of Joule and others have proved this; the visible kinetic energy of the moving bodies has been changed into the invisible energy of the molecules of those bodies. When the stone falls and strikes the ground, the total energy of the earth and stone remains unchanged; when the two bodies impinge and are brought to rest, they are heated by the impact and the heat is energy equal in amount to the kinetic energy of the masses.

Energy may take other forms besides the potential and kinetic energy of bodies sufficiently large for us to see. The total amount of energy existing in two or more bodies cannot be altered by any mutual action between those bodies[1].

122. Conservation of Energy. It was said above that in many cases of motion the sum of the kinetic and potential energies of the system considered remains the same; we have now been led to a wider generalization as the result of observation and experiment. We may in Maxwell's words state it thus.

PRINCIPLE OF THE CONSERVATION OF ENERGY. *The total energy of any material system is a quantity which can neither be increased nor diminished by any action between the parts of the system though it may be transformed into any of the forms of which energy is susceptible.*

123. Conservation of Energy in Mechanics. When stated as above the principle of the conservation of

[1] For an account of some of the experiments necessary to prove the statements made above, see Glazebrook, *Heat*, Chaps. I. and XIII.

energy is too wide for effective use in **Mechanics**, it includes the whole of **Physical Science**. We can however limit it and put it into a form which will be of assistance to us. Now we have seen that there are some cases of motion in which there is no change in the sum of the kinetic and potential energies, while in other cases energy is dissipated as heat or in some other form. In the first case the system considered is said to be a **Conservative System**; when dealing in Mechanics with a conservative system the two forms of energy with which we are concerned are kinetic and potential; the sum of these two forms is always the same. The gain of kinetic energy in any change is equal to the loss of potential energy during the same change, and vice versa.

DEFINITION. *In Mechanics a system is said to be* **Conservative** *when the amount of work necessary to bring it from any one condition to any other is always the same and does not depend upon the steps by which that change is carried out.*

For example, the same amount of work is necessary to raise a body from one given position to another given position, by whatever path the body be raised, provided that we are concerned only with the mutual action between the Earth and the body, and the constraints introduced by smooth surfaces.

This system then is a conservative system. It can in fact be shewn that if the impressed force or rate of change of momentum of each part of the system depends only on the position of that part relative to the other parts and not on its velocity, then the system is conservative.

A body sliding down a rough surface is losing momentum owing to friction at a rate which depends partly on its speed and on the direction in which it is going; when *sliding down*, it gains momentum from the action of the Earth but loses it owing to friction; when *sliding up*, both actions contribute to the loss of momentum; this system is not conservative.

In order that the principle of the conservation of energy may be of use to us in solving mechanical problems—in which we deal only with the kinetic and potential energies of visible bodies—it is necessary that the system considered should be a conservative one.

In a Conservative system the sum of the kinetic and potential energies of the system can only be changed by action exercised on the system from without.

Assuming then this principle as established by reasoning of a general character from the fundamental laws and definitions, it may be applied to the solution of individual problems.

Thus the system consisting of the Earth and a heavy body moving on a smooth surface is a conservative one, the potential energy depends only on the height z above the surface of the Earth; the kinetic energy is $\frac{1}{2}mv^2$; if the body start from rest at a height h we have

$$\tfrac{1}{2}mv^2 + mgz = mgh,$$

or

$$v^2 = 2g(h - z).$$

Again, the mutual potential energy of two masses m, m_1, for which Newton's law of gravitation holds, can be proved to be mm_1/r, where r is their distance apart. Thus if v, v_1 be their velocities, their total energy is

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}mv_1^2 + mm_1/r.$$

If the bodies move to a distance r' apart at which they have velocities v', v_1', then the energy is

$$\tfrac{1}{2}mv'^2 + \tfrac{1}{2}m_1v_1'^2 + mm_1/r',$$

and these two expressions for the energy are equal.

Thus

$$\tfrac{1}{2}m(v^2 - v'^2) + \tfrac{1}{2}m_1(v_1^2 - v_1'^2) = \frac{mm_1}{r'} - \frac{mm_1}{r}.$$

124. Unit of Energy. Since energy is measured as work the unit of work is the **Unit of Energy**, its actual value then will depend on the units we use for length, time and mass. On the C.G.S. absolute system the unit of energy is the **Erg**; if we are measuring force in grammes' weight, the unit of energy is the **Centimetre-gramme.**

If again we are working in feet and pounds we have on the absolute system as unit of energy the **Foot-poundal**, and in gravitation units the **Foot-pound.**

When however it is stated that the kinetic energy of a moving mass is $\frac{1}{2}mv^2$, the truth of the relation $F = ma$ has

been assumed; this implies that on the C.G.S. system the force is measured in Dynes, the energy therefore is in Centimetre-Dynes or Ergs, while on the F.P.S. system the energy is in Foot-poundals. *In the various statements made in the preceding sections, it is of course assumed that a consistent system of units is employed throughout.*

Thus on the C.G.S. system the statement that the kinetic energy is $\frac{1}{2}mv^2$ means that it is $\frac{1}{2}mv^2$ ergs; if we wish to express it in centimetre-grammes we must remember that 1 dyne is $1/g$ (or $\frac{1}{981}$) of the weight of a gramme.

Hence in centimetre-grammes the kinetic energy is $\frac{1}{2}mv^2/g$ or $\frac{1}{2}mv^2/981$.

Again the kinetic energy of a mass of m pounds moving with a velocity of v feet per second is $\frac{1}{2}mv^2$ foot-poundals or $\frac{1}{2}mv^2/g$ or $\frac{1}{2}mv^2/32{\cdot}2$ foot-pounds.

125. Energy, Momentum and Force. We have thus been led to deal with two quantities,—**Energy** and **Momentum**,—depending on the motion of bodies: each of these is unchanged in total amount by mutual action between the bodies which make up the system, each can be transferred from one body of the system to another. Energy may exist in various forms, it may change from one form to the other in the course of the motion, but all these forms can be measured in terms of one common unit, and when so measured their sum total remains the same. Momentum we only know in the form of the product of the mass of a body and its velocity.

Momentum and kinetic energy are closely connected; to measure the kinetic energy of a particle we multiply its momentum by its velocity and divide by 2.

Force stands in a different category to these two quantities, its amount does not in general remain unchanged during the motion; we find however that the conception of force is useful to connect together momentum and energy, and to express certain observed facts.

Thus suppose a body of mass m is observed to move from rest with a uniform acceleration a, and to describe a distance s in time t. Then we find that the following three quantities —mv/t, ma and $\frac{1}{2}mv^2/s$—each of which we can determine by observation are equal.

Each of these may be defined to be the impressed force: calling it F we have

$$F = \frac{mv}{t} = ma = \frac{\frac{1}{2}mv^2}{s}.$$

If the initial velocity be u the formulæ become

$$F = \frac{m(v-u)}{t} = ma = \frac{\frac{1}{2}mv^2 - \frac{1}{2}mu^2}{s}.$$

We also deal with the products Ft which measures the whole change of momentum or the Impulse, and Fs which measures the Work or whole increase of kinetic energy.

We find moreover that in a large number of cases these quantities depend only on the position of the body with reference to surrounding objects and are quite independent of its velocity or direction of motion. To each of these quantities the name of Force is given. These results are obtained from the observation of certain simple cases of motion. They are then generalized and by their aid the motion of bodies under very complex circumstances can be calculated.

Newton founded the Science of Dynamics on the first two of the above relations. Force he defines as rate of change of momentum. In the Corollary to the third law he draws attention to the importance of the last relation and emphasizes some of its principles. In his view it follows as a mathematical consequence of the first relation; it might of course have been taken as the starting point of the subject, basing it as has been done in the preceding pages on the fact that a body moving along a smooth curve will rise to the same height as that from which it started; this indeed had been done by Galileo, and it was by this method that Huyghens had obtained his results about the motion of pendulums.

***126. Graphical construction for Work.** When the impressed force is uniform the work done is Fs; when it is variable we suppose it to be uniform while the body moves over a number of very short spaces s_1, s_2 etc., but to change at the end of each of these spaces. We can express this by a graphical construction identical with that used in Section 41 to determine the space traversed in terms of the time.

PROPOSITION 32. *To determine graphically the work done by a force.*

Draw a line OX, Fig. 77, to represent space and let NN' be the distance traversed under a force F. Draw NP and $N'P'$ at right angles to NN' to represent the force and join PP'. Then the area $PNN'P'$ is equal to Fs. But Fs is the work done on the body when moving the distance s; hence the area $PNN'P'$ represents the work.

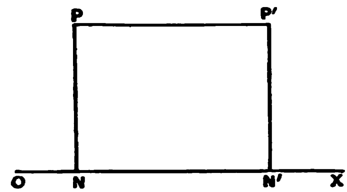

Fig. 77.

If the force be not constant but change at the ends of the spaces s_1, s_2... etc., from F_1 to F_2, F_2 to F_3, etc., respectively, the work done will be represented by the area consisting of a number of rectangles such as $N_1P_1R_1N_2$, $N_2P_2R_2N_3$, etc., Fig. 78.

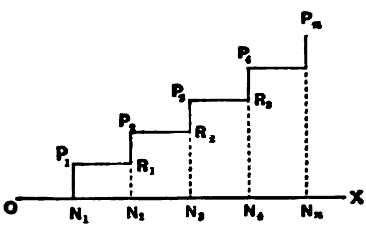

Fig. 78.

By diminishing the spaces N_1N_2, N_2N_3, etc., during which we deal with the force as uniform, we get the case of a varying force; the broken line $P_1R_1P_2$ ... of Fig. 78 becomes a

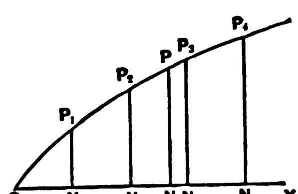

Fig. 79.

continuous curve P_1PP_4, Fig. 79, and the area $P_1N_1N_4P_4$ represents the work done by traversing the distance N_1N_4. Thus if we construct a figure on squared paper in which the horizontal divisions represent space, and the vertical divisions force, by drawing a curve P_1P' such that the line PN perpendicular to the space line may represent the force when the body has traversed a space N_1N, the area $P_1N_1N'P'$ represents the work done during the motion. In such a diagram if the horizontal divisions represent centimetres and the vertical divisions be dynes, then the work in ergs is given by the number of squares contained within the area.

The following is an example of the method.

PROPOSITION 33. *To calculate the work done in stretching a spiral spring.*

Let ON_1, Fig. 80, be the original length of the spring and suppose it stretched so that its length is ON. Then N_1N represents the extension. Now we have seen, Section 91, that the force required to extend a spring is proportional to the extension. Thus if PN drawn vertical at N represent the force which will just hold the spring extended to the length ON, we see that PN is always proportional to N_1N: if we wish to extend the spring by twice N_1N the force necessary to hold it in this position will be twice PN. Thus the curve corresponding, Fig. 79, which gives the force in terms of the displacement, is a straight line through the points P and N_1, and the area which determines the work is a triangle.

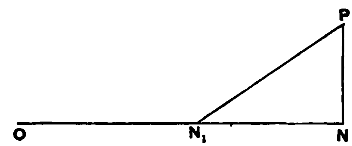

Fig. 80.

Thus the work done in extending the spring from N_1 to N is the area of the triangle PNN_1.

Hence if we call F the force, and s the final extension produced under this force, we see that

$$\text{Work done} = \text{Area } PNN_1$$
$$= \tfrac{1}{2}PN \, . \, NN_1 = \tfrac{1}{2}Fs.$$

Examples. (1). *Find the energy of a mass of* 1 *cwt. while falling from a height of* 100 *feet.*

The total energy at any point of the fall is equal to the potential energy at the starting-point, and this is equal to the work done in raising the body from the ground. This work has been shewn Example 1, p. 160, to be 15190 Joules.

(2). *Compare* (*a*) *the momenta,* (*b*) *the kinetic energies of a bullet whose mass is* 100 *grammes moving with a speed of* 400 *metres per second, and a cannon-ball whose mass is* 50 *kilogrammes moving with a speed of* 10 *metres per second.*

Reduce the speeds to centimetres per second and the masses to grammes, then we have

(*a*) Momentum of the bullet $= 100 \times 400 \times 100$
$= 4 \times 10^6$ C.G.S. units of momentum.

Momentum of cannon-ball $= 50 \times 1000 \times 10 \times 100$
$= 5 \times 10^7$ C.G.S. units of momentum.

(*b*) Energy of bullet $= \frac{1}{2} 100 \times 16 \times 10^8$
$= 8 \times 10^{10}$ ergs.

Energy of cannon-ball $= \frac{1}{2} 50 \times 1000 \times 1 \times 10^6$
$= 2 \cdot 5 \times 10^{10}$ ergs.

Thus the cannon-ball has the greater momentum while the bullet has the greater energy.

(3). *The bullet and the cannon-ball are each brought to rest with uniform retardation in* 1 *second. Determine the impressed forces and the distance each moves.*

In the case of the bullet 4×10^6 units of momentum are destroyed in one second, thus the impressed force is 4×10^6 dynes and the retardation is $4 \times 10^6/100$ or 4000 cm. per sec. per sec.

The distance the bullet will travel while being stopped in one second is thus $\frac{1}{2}$ (4000) or 2000 centimetres.

For the cannon-ball 5×10^7 units of momentum are destroyed in one second, thus the impressed force is 5×10^7 dynes, and since the mass is 50000 grammes the retardation is $5 \times 10^7/5 \times 10^4$ or 1000 cm. per sec. per sec. The distance the cannon-ball moves while being stopped in 1 second is thus 500 centimetres.

Thus if both bodies are stopped in the same time and both lose their momentum at uniform rates, the rate of loss for the cannon-ball is $\frac{50}{4}$ or 12·5 times as great as that for the bullet, but the bullet moves $\frac{20}{5}$ or 4 times as far as the cannon-ball.

(4). *A bullet* 100 *grammes in mass is fired from a gun the barrel of which is* 75 *cm. long and leaves it with a velocity of* 400 *metres per second; assuming the pressure due to the powder to be uniform, find the impressed force on the bullet and the time it takes to traverse the barrel.*

Let F dynes be the impressed force. The kinetic energy of the bullet is $\frac{1}{2}100 \times (40000)^2$, or 8×10^{10} ergs.

It acquires this energy while moving through 75 centimetres. Hence

$$F \times 75 = 8 \times 10^{10}.$$

$$F = \frac{8 \times 4 \times 10^{10}}{3 \times 10^2} = \frac{32}{3} \times 10^8 \text{ dynes.}$$

The momentum of the bullet is 4×10^6 C. G. S. units. The time during which the bullet is in the barrel is given by dividing this by the impressed force; let it be t seconds.

Then $$t = \frac{4 \times 10^6 \times 3}{32 \times 10^8} = \frac{3}{8} \times 10^{-2},$$

or three eight-hundredths of a second.

(5). *An engine developes* 5000 *horse-power while driving a ship at the rate of* 25 *miles an hour. Find the resistance offered to the motion.*

The energy supplied by the engine is employed in doing work against the resistance. The velocity of the ship is $36\frac{2}{3}$ feet per second, the rate at which work is being done is 550×5000 foot-pounds per second, and this is equal to the resistance multiplied by the velocity. Hence if R represent the resistance in lb. weight

$$R \times 36\tfrac{2}{3} = 550 \times 5000,$$

$$R = \frac{550 \times 5000 \times 3}{110} = 75000 \text{ lb. weight.}$$

(6). *A simple pendulum, the mass of which is* 1 *kilogramme, is started from its lowest point with a velocity of* 120 *cm. per second; the pendulum makes one oscillation per second and loses energy from friction and other causes at the rate of* 1 *centimetre-gramme unit per second. Determine for how long it will continue to move.*

The kinetic energy of the pendulum is

$$\tfrac{1}{2} \times 1000 \times 14400 \text{ ergs,}$$

or $$\tfrac{1}{2} \times \frac{1000 \times 14400}{981} \text{ centimetre-gramme units.}$$

This reduces to 7339 centimetre-gramme units of energy. Since 1 of these units is lost each second, the pendulum will continue to move for 7339 seconds or for 2 hours 2 minutes 19 seconds.

EXAMPLES.

WORK AND ENERGY.

1. Define Energy, and explain how to measure (*a*) the energy of a bullet as it leaves the muzzle of a gun, (*b*) the energy of a clock pendulum at the highest and lowest points of its swing.

2. State the principle of the Conservation of Energy as employed in Mechanics, and illustrate it by some examples.

By the use of this principle shew that the velocity acquired by a body falling from rest down a smooth inclined plane depends only upon the vertical height of the plane and not upon its length.

3. Distinguish between work and power. A watt is equivalent to 10^7 ergs per second; the acceleration of gravity is 981 cm. per sec. per sec.; find how long a kilogramme has been falling from rest when gravity is doing work upon it at the rate of one watt.

4. Calculate the momentum and the energy of (1) a bullet weighing $\frac{1}{2}$ an oz. moving at the rate of 1200 feet per second, (2) a mass of $\frac{1}{2}$ a ton moving at the rate of 6 inches per second. Find the forces required to stop the two in $\frac{1}{10}$ second and the work which each would do in being stopped.

5. If a body be moving under the action of a constant force, shew that the horse-power developed by the force is proportional to the force and to the velocity of the body.

6. Distinguish between kinetic and potential energy.

A pendulum consisting of a ten-gramme bob at the end of a string thirty centimetres long oscillates through a semi-circle; find its kinetic energy when the string makes an angle of 45° with the vertical.

Specify the units in which your answer is expressed.

7. Shew how the second law of motion enables us to measure force and mass. What do you understand by "action" in the statement of the third law? Illustrate your answer by some applications of the law.

8. A mass of 50 grammes moving with a velocity of 12 cm. per second overtakes and adheres to a mass of 30 grammes moving with a velocity of 4 cm. per second. Find the common velocity and calculate the total kinetic energy before and after the impact.

9. A mass of 1 cwt. is moving with a velocity of 1 foot per second. Determine the velocity of a bullet whose mass is 1 oz. when it has (1) the same momentum, and (2) the same kinetic energy as the larger mass.

10. A force equal to the weight of 10 lb. acts for a minute on a mass of 1 cwt. Find the momentum and energy of the mass. What is the work done by the force?

11. A bullet whose mass is 1 oz. leaves the muzzle of a gun with the velocity of 1000 feet per second, find its energy in foot-poundals; and if the length of the barrel of the gun is 3 feet, find the mean pressure exerted by the powder on the bullet.

12. A bullet whose mass is an ounce moving with a velocity of 2400 feet per second strikes a block of wood at rest and remains imbedded in the wood: if the resulting velocity of the block and bullet together be 16 feet per second, calculate the mass of the wood. Also calculate the energy lost when the bullet penetrates the wood, and express the result in foot-tons.

13. A man whose mass is 150 lb. walks up a hill of 1 in 6 at the rate of 4 miles an hour; what fraction of a horse-power is he doing?

14. Find the amount of work done in pushing a mass of 10 lb. through 5 feet up an incline of 1 in 10, neglecting friction.

15. Find the amount of horse-power transmitted by a rope passing over a wheel 10 feet in diameter which makes 1 revolution per second, the tension in the rope being 100 lbs.

16. Point out the transformations of energy that take place during the swinging of a pendulum. State at what point of its swing a pendulum must be if its energy is half potential and half kinetic.

17. How much energy has a mass of 1 cwt. when moving at the rate of 100 yds. per sec.? In what units is your answer expressed?

18. A man can bicycle 12 miles an hour on a smooth road. He exerts a downward pressure of 20 lb.-wt. with each foot during the down-stroke, and the length of down-stroke is 12 inches. His driving wheel is 12 feet in circumference. Find the work he does per minute.

19. A shot whose mass is half a ton is fired with a velocity of 2000 feet a second from a gun whose mass is 50 tons; neglecting the weight of the powder, find the velocity with which the gun will recoil, if mounted so that it moves without friction along a level tramway.

Compare the work done on the gun with that done on the shot.

20. A cannon weighs 35 tons, and the shot half a ton. The velocity of the shot on leaving the muzzle is 1200 ft. per second; find the velocity of the recoil of the cannon. Neglect the inertia of the gases formed by the burning of the powder.

Will the effect of these gases be to reduce or to increase the recoil? Give your reasons.

21. What is the horse-power required to fill in 3 hours a tank 9 feet deep, 20 feet long, and 10 feet wide, placed on the top of a building 60 feet in height, from a well in which the surface of the water remains constantly 24 feet below the ground level? Give the answer correct to two places of decimals.

(Mass of cubic foot of water $= 62\frac{1}{2}$ lb.)

22. Assuming that the resistance of a train moving along a horizontal railway at 35 miles per hour is equivalent to an incline of 1 in 280 find the horse-power required to take a train of 100 tons along a horizontal railway at a rate of 35 miles per hour.

23. The mass of a ship is 3000 tons; assuming that the resistance to its motion varies as the square of the speed and that the force required to give it a speed of 1 foot per second is equal to $\frac{1}{150000}$ the weight of the ship, find the horse-power requisite to propel it at the rate of 30 feet per second.

24. Define the kinetic energy and the potential energy of a system.

A fine string passes through two small fixed rings, A and B in the same horizontal plane, and carries equal weights at its ends, hanging freely from A and B. If a third equal weight is attached to the middle point of the portion AB of the string, and is let go, prove that it will descend to a depth equal to two-thirds of the length AB below AB, and will then ascend again.

25. A mass of 4 cwt. falls from a height of 10 feet on to an inelastic pile of 12 cwt. Supposing the mean resistance to the penetration of the pile to be $1\frac{1}{2}$ tons weight, determine the distance it is driven at each blow.

26. A smooth wedge of mass M and angle α rests upon a horizontal plane; another mass m is placed upon its slant surface, and the system begins to move.

Write down (1) the equation of energy, (2) the equation of linear momentum.

27. I strike an anvil with a given hammer, and its velocity on reaching the anvil is the same as that with which it reaches a piece of red-hot iron on the anvil. Will the impulse be the same in the two cases?

What quantities must be known in order to compare impulses?

28. A bullet weighing 1 oz. leaves the mouth of a rifle whose barrel is 4 feet long with the velocity of 1000 ft. per second. Find the mean force on the bullet, neglecting the friction against the sides of the barrel.

29. A cannon-ball whose mass is 1 cwt. moving with a velocity of 25 yds. per sec. penetrates to a depth of 10 feet into a sandbank. Find the average pressure on the sand.

30. The erg and foot-pound are both units of work: a horse-power is 33,000 foot-pounds per minute; how many ergs per hour would this be?

[1 inch = 2·54 centimetres. 1 lb. = 453·6 grammes.]

31. In a system of distributing power by means of water at a high pressure, the pressure of water is 2000 lb.-wt. per square inch. How many cubic feet must be used per hour to supply 10 H.-P. (1 H.-P. = 33000 foot-pounds of work per minute) assuming no power to be lost?

32. A mass of m pounds is raised up a plane inclined at an angle of 30° to the horizon, and of length l, and reaches the top with a velocity v. Shew that the work done is

$$\frac{m}{2}\left(1+\frac{v^2}{g}\right) \text{ foot-pounds.}$$

*CHAPTER IX.

CURVILINEAR MOTION UNDER GRAVITY.

127. Projectiles. When a ball is thrown in the air it does not move in a straight line; a very little observation is sufficient to shew this; moreover its velocity is continually changing. We can deduce the form of the path from the laws of motion, thus—Let us suppose, in order to simplify the problem, that the body is projected in a horizontal direction with a velocity u; the only acceleration which it acquires is g in a vertical direction due to the action of the Earth, and it is not difficult to determine the path of the body under these circumstances. We will however in the first instance investigate the motion by the aid of experiment.

We have seen that two bodies whatever be their masses when dropped together fall at the same rate.

We wish now to shew that, if one of the bodies be projected in a horizontal direction with any velocity while the other is allowed to drop simultaneously from the same height, the two will fall at the same rate and reach the ground together. We may shew this roughly by rolling a ball rapidly along a table; on leaving the table it will describe a curve in the air; if now at the moment the first ball leaves the table a second be dropped from the same height the two will reach the ground together. The same fact is better shewn by the aid of an arrangement of apparatus devised by Sir Robert Ball[1].

[1] *Experimental Mechanics*, Section 511.

EXPERIMENT 24. *To shew that the time of fall of a body is independent of its horizontal velocity.*

In Fig. 81, AB is a piece of wood about 2·5 cm. thick the upper edge of which is curved as shewn. A strip of thin

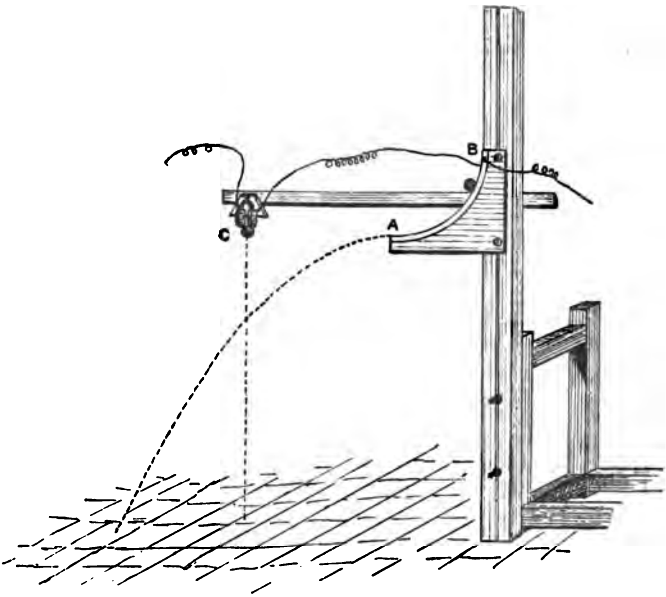

Fig. 81.

brass is screwed to each face of the board, the edges of the brass strips being also curved: care must be taken that there shall be no metallic connection between the brass strips. Thus the upper edge of the wooden board forms a kind of groove with brass sides. The brass strips are connected to binding screws. On resting a brass ball on the strips as shewn in the figure, an electric current can pass from one strip to the other through the ball.

One binding screw is connected to a battery, the other to the electromagnet described in Section 65; a wire also

passes from the electro-magnet to the second pole of the battery. When the ball is on the groove the circuit is complete and the magnet is made; thus it can support a small iron ball.

When the brass ball rolls off the groove the circuit is broken between the strips and at the same moment the iron ball drops. If the groove be fixed in such a position that its direction at A is horizontal the brass ball is projected in a horizontal direction.

If also the electromagnet C be fixed at the same height as the point of projection A, then the iron ball is dropped and the brass ball projected horizontally at the same moment from the same height. The velocity with which the brass ball starts can be varied by allowing it to roll[1] down the groove AB from various positions, or by placing a spring in the groove and projecting the ball forward by its aid. If this plan be adopted a straight horizontal groove will do as well as the curved one shewn.

Arrange the apparatus as described and, starting the brass ball from various points in the groove, observe the times at which the two balls reach the floor. It will be found that whatever be the height of the starting point and whatever be the velocity of projection the two always strike the floor simultaneously.

Thus the downward acceleration of the two balls is the same; the brass ball although it starts with a horizontal velocity, depending on the distance it has rolled down the groove, gains in each second the same vertical velocity as the iron ball; in the first second it will fall through $\frac{1}{2}g$ centimetres, in the second through $\frac{4}{2}g$ centimetres, and in t seconds $\frac{1}{2}gt^2$ centimetres. The vertical velocity is independent of the horizontal and is the same as that of a body allowed to drop freely.

[1] If the ball is not quite spherical then in rolling down it may happen that contact between the ball and the strip is broken, and the iron ball is allowed to drop too soon. This may be avoided by using instead of a ball a cylindrical piece of brass or zinc which slides down on the brass strips; the friction however in this case is greater and the ball does not start with so great a velocity.

EXPERIMENT 25. *To describe a parabola.*

Take a straight lath or scale about a metre long; divide it into equal distances each 10 cm. in length and from each point of division suspend by a piece of fine string or thread a small bullet or weight.

Adjust the first piece of string to some convenient length, say 3 centimetres, make the second 3×2^2 cm., the third 3×3^2 cm., the fourth 3×4^2 and so on, so that the lengths of any two strings are proportional to the squares of their distances from the end of the rod.

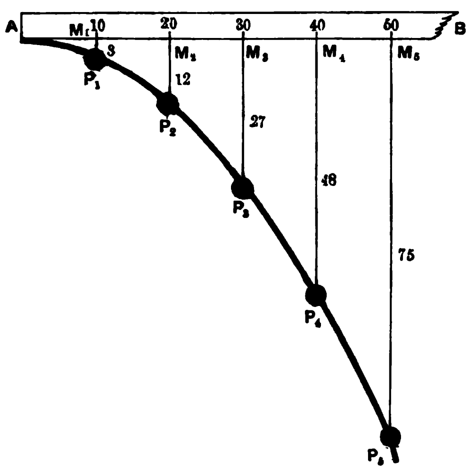

Fig. 82.

Thus, in Fig. 82, AB is the lath, M_1, M_2, M_3, etc. the points of division, P_1, P_2 etc. the bullets, and we have $P_1M_1 = 3$ cm.,

$$P_2M_2 = 3 \times 2^2 = 12 \text{ cm.},$$
$$P_3M_3 = 3 \times 3^2 = 27 \text{ cm.},$$

and so on.

Hold the lath as shewn in Fig. 82 against a vertical blackboard or sheet of drawing-paper, mark the positions P_1, P_2 etc.

of the bullets. The points so found lie on a curve called a parabola. If we suppose that each of the 10-centimetre divisions is subdivided into (say) 10 parts, and that threads with bullets are hung from these in such a way that the lengths of the threads may follow the same law as before, the bullets will be practically in contact and the curve which passes through them will be a parabola. The curve however can be constructed with sufficient accuracy for our purposes by drawing a free-hand curve through the points $P_1 P_2 \ldots$

Again, draw a vertical line AN as in Fig. 83 from A, the end of the lath from which the divisions are reckoned, and from P_1 draw P_1N_1 parallel to the lath to meet AN in N_1.

Then
$$P_1N_1 = AM_1,$$
$$AN_1 = PM_1,$$

and by construction PM_1 is proportional to AM^2.

Hence $P_1N_1^2$ is proportional to AN_1.

In the figure the lath is horizontal, this however is not necessary; in whatever direction the lath be held the balls still lie on a parabola; the size of the curve will depend on the inclination of the lath.

EXPERIMENT 26. *To determine the path of a body projected in a horizontal direction and to shew that it is a parabola.*

Arrange the grooved board described in Experiment 24 in front of a vertical black-board as shewn in Fig. 83 in such a way that the ball after sliding down the groove may start from A in a horizontal direction and fall in a vertical plane parallel to the board. Make a mark C on the groove and in all the experiments start the ball from this mark. Allow the ball to roll down the groove and watch its path.

Fix to the board with drawing-pins a number of paper hoops so that the ball in its path passes through each of them; the proper position of the uppermost hoop is first found by trial, then that of the next below, and so on, the ball being started in each case from the same point C. Mark on the board the positions of the centres of the hoops, remove them and draw with a free hand a curve starting from A and passing through the various marks $P_1, P_2 \ldots$. Draw a horizontal line from A, and vertical lines P_1M_1, $P_2M_2 \ldots$ from the marks

to meet it in M_1, M_2. Measure the horizontal distances AM_1, AM_2..., and the vertical distances P_1M_1, P_2M_2, etc.

Write down the squares of AM_1, AM_2 etc., and obtain the quotients, given by dividing each square such as AM_1^2, by

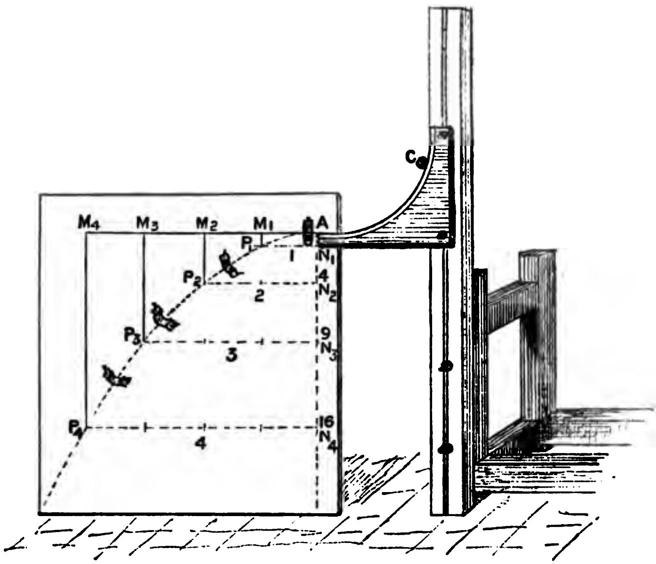

Fig. 83.

the corresponding vertical distance; it will be found that these quotients are all approximately equal. But the curve which has this property is the parabola. Thus the path is a parabola.

Again, measure the vertical height above A of the point C from which the ball starts; let it be a. Then it will be found that the constant ratio of AM^2 to PM is equal to $4a$.

Thus we have

$$AM^2 = 4aPM.$$

But if u be the horizontal velocity with which the ball leaves the groove we have seen that u is acquired by falling down a height a, hence $u^2 = 2ga$.

Therefore $$a = u^2/2g,$$

and $$AM^2 = \frac{2u^2}{g} PM.$$

The curve described by P is a parabola, the point A is called the vertex of the parabola, and the quantity $2u^2/g$ is its Latus Rectum.

Thus, when a body is projected in a horizontal direction its path is a **Parabola.**

Again, if we suppose the motion of the body at any point to be reversed, it will proceed to describe the same path in the reverse direction; thus when projected obliquely its path will still be a parabola. This may be verified by construction in a similar way by arranging the grooved board so that its direction is not horizontal.

The path of the drops of water in a water-jet is the same as that of a body projected under gravity, each little particle of water follows the same course as that which would be taken by the body if projected with the velocity with which it starts. By placing a lamp at some distance from such a jet its shadow can be thrown on a white screen placed behind it, the path of the jet can thus be traced and measurements made on it as on the curve drawn as described above. For further details see Glazebrook and Shaw, *Practical Physics*, Section C.

128. Motion of a Particle projected under Gravity. We have shewn by the result of Experiment 24 that the vertical motion of a falling body is independent of its horizontal velocity. A body which has initially no vertical velocity will in t seconds fall a distance $\frac{1}{2}gt^2$, whether it start from rest or be projected horizontally.

If the body be projected obliquely, its velocity has a vertical as well as a horizontal component; let v be the upward vertical component, u the horizontal component.

Then during t seconds the velocity v will have carried the body a distance vt upwards, while owing to the vertical acceleration g the displacement will be $\frac{1}{2}gt^2$ downwards. Thus h, the actual height above the point of projection, is given by

$$h = vt - \tfrac{1}{2}gt^2.$$

In the same time the horizontal velocity u will have carried it a distance ut, and it has no horizontal acceleration, hence k the horizontal displacement is given by

$$k = ut.$$

If the actual velocity of projection be U and the direction of motion be inclined at an angle α to the horizon, then we have

$$u = U \cos \alpha,$$
$$v = U \sin \alpha.$$

Hence

$$h = Ut \sin \alpha - \tfrac{1}{2}gt^2,$$
$$k = Ut \cos \alpha.$$

We will now shew how to prove that the path is a parabola so long as the resistance of the air is neglected.

PROPOSITION 34. *A particle is projected with a given velocity and at a given inclination to the horizon; to shew that its path is a parabola.*

Let P, Fig. 84, be the point of projection and PT, making an angle α with the horizontal line Px, the direction of projection. Let U be the velocity of projection along PT.

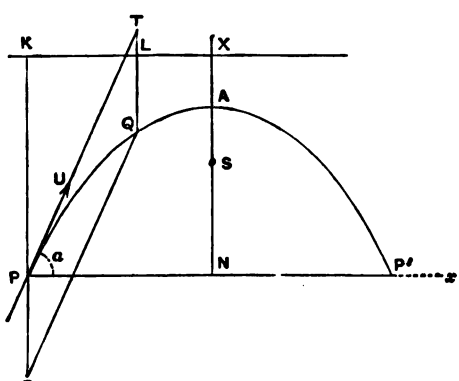

Fig. 84.

The only acceleration which the particle has is vertical and is equal to g. The motion will therefore take place in the vertical plane through PT.

To find the position of the particle after t seconds, make PT equal to Ut and from T draw TQ vertically downwards and equal to $\frac{1}{2}gt^2$. Draw PR vertical and equal to TQ and join QR. Then if the particle had no acceleration it would move uniformly along PT and at the end of t seconds would be at T. Again, if it had initially no velocity it would in t seconds fall vertically and reach the point R. In the actual circumstances the displacement in each of these two directions is, in accordance with the second law, independent of that in the other, the particle is displaced in the direction PT just as far as it would be if it had no vertical acceleration; it is displaced vertically just as far as it would be if it had initially no velocity along PT. Thus, at the end of t seconds, PT still represents the displacement due to the initial velocity, while PR or TQ represent the displacement due to the acceleration. Hence the particle is at Q.

Now $$QT = \tfrac{1}{2}gt^2,$$
$$PT = Ut.$$

Hence $$t^2 = \frac{2QT}{g}, \quad t = \frac{PT}{U},$$

and $$2\frac{QT}{g} = t^2 = \frac{PT^2}{U^2}.$$

Therefore $$PT^2 = \frac{2U^2}{g} . QT.$$

Now U and g are both constant, therefore the ratio of PT^2 to QT is a constant, and this (Experiment 25) is the fundamental property of a parabola.

Thus the point Q always lies on a parabola.

129. Properties of the path of a Projectile. We will now proceed to investigate various properties of the motion of a projectile.

(1) *To find the directrix of the parabola.*

Draw PK vertical and equal to $U^2/2g$. Then the velocity at P would be acquired by falling through a height KP. A horizontal line KX drawn

through K is called the directrix. The velocity at any point of the parabola is that due to the fall from the directrix.

For if PK is equal to H, and if h be the height above P of the particle when at Q, since the energy remains constant we have, if V denote the velocity at Q,

$$\tfrac{1}{2}mV^2 + mgh = \tfrac{1}{2}mU^2 = mgH.$$

Hence $V^2 = 2g(H-h) = 2gQL$ if the vertical QT meet the directrix in L.

(2) *To find the vertical and horizontal components of the velocity at any time.*

The horizontal velocity initially is $U\cos\alpha$, and since there is no horizontal acceleration it remains unchanged.

The vertical velocity initially is $U\sin\alpha$, and in t seconds under the downward vertical acceleration g an additional vertical velocity $-gt$ is acquired.

Hence if u, v represent the components of the velocity at any time

$$u = U\cos\alpha,$$
$$v = U\sin\alpha - gt.$$

(3) *To find the direction of motion at any time.*

If at any time t the particle be moving with velocity V in a direction making an angle θ with the horizon, we have

$$V\cos\theta = u = U\cos\alpha,$$
$$V\sin\theta = v = U\sin\alpha - gt.$$

Hence
$$V^2 = U^2\cos^2\alpha + (U\sin\alpha - gt)^2$$
$$= U^2 + g^2t^2 - 2Ugt\sin\alpha,$$
$$\tan\theta = \frac{U\sin\alpha - gt}{U\cos\alpha}.$$

(4) *To find the position of the particle at any time.*

Let h be the distance of the particle above P, k its horizontal distance from P.

Then
$$h = Ut\sin\alpha - \tfrac{1}{2}gt^2,$$
$$k = Ut\cos\alpha.$$

(5) *To find the time to the vertex.*

At the vertex A (Fig. 84) the motion is horizontal, the vertical velocity therefore is zero. Hence if t_1 is the time to the vertex

$$U\sin\alpha - gt_1 = 0,$$
$$t_1 = \frac{U\sin\alpha}{g}.$$

(6) *To find the height of the vertex.*

If the height of the vertex AN (Fig. 84) be h_1, then h_1 is the height of the particle at time t_1 when the vertical velocity is zero.

Hence $$h_1 = Ut_1 \sin \alpha - \tfrac{1}{2}gt_1^2.$$

But $$t_1 = \frac{U \sin \alpha}{g}.$$

Therefore $$h_1 = \tfrac{1}{2}\frac{U^2 \sin^2 \alpha}{g}.$$

(7) *To find the distance of the vertex from the directrix.*

At the vertex the velocity is horizontal and is equal to $u \cos \alpha$.

Hence if AX be perpendicular from A on the vertex, we have

$$AX = \frac{U^2 \cos^2 \alpha}{2g}.$$

In a parabola four times the distance between the vertex and the directrix is the latus rectum. Hence the latus rectum is $2U^2 \cos^2 \alpha/g$.

In Section 125 we found the value $2u^2/g$ for the latus rectum; it must be remembered that u is the constant horizontal velocity which is equal to $U \cos \alpha$. Thus the two formulæ are the same.

A line through the vertex at right angles to the directrix is called the axis of the parabola.

A point S on this line at a distance from the vertex equal to AX is called the focus.

(8) *To find the horizontal distance between the vertex and the point of projection.*

Let the distance PN (Fig. 84) be k_1. Then k_1 represents the horizontal distance which the particle has moved in time t_1.

Hence $$k_1 = U \cos \alpha t_1 = \frac{U^2 \cos \alpha \sin \alpha}{g} = u\frac{U \sin \alpha}{g},$$

writing u for the constant horizontal velocity $U \cos \alpha$.

Moreover $$h_1 = \frac{U^2 \sin^2 \alpha}{2g}.$$

Hence $$k_1^2 = \frac{2u^2}{g}\,\frac{U^2 \sin^2 \alpha}{2g}$$
$$= \frac{2u^2}{g} h_1.$$

Thus $$PN^2 = 4AX \,.\, AN,$$

and the ratio of PN^2 to AN is the same for all points. (Cf. Section 125.)

(9) *To find the time at which the particle reaches the ground again.*

Let the time be t_2.

Then at time t_2 the height h of the particle is zero.

Hence $$0 = Ut_2 \sin \alpha - \tfrac{1}{2} g t_2^2.$$

Therefore $t_2 = 0$, which gives the starting-point, or

$$t_2 = \frac{2U \sin \alpha}{g} = 2t_1,$$

which gives the time to the point P'.

Thus the particle takes as long to descend from A to P' as to rise from P to A.

The value of t_2 is known as the time of flight.

(10) *To find the range on the horizontal plane.*

The range PP' is the horizontal distance which the particle moves in time t_2.

Hence $$\text{Range} = Ut_2 \cos \alpha = \frac{2U^2 \sin \alpha \cos \alpha}{g}$$
$$= \frac{2u}{g} . U \sin \alpha = \frac{U^2 \sin 2\alpha}{g}.$$

Hence for a given velocity of projection the range is greatest if $\sin 2\alpha$ is greatest, that is if $2\alpha = 90°$, $\alpha = 45°$.

A number of other propositions on parabolic motion might be given; for these the reader is referred to Loney, *Elementary Dynamics*.

Examples. (1). *Shew by finding an expression for the velocity of a projectile that it is equal to that due to a fall from the directrix.*

If H be the distance between the directrix and the point of projection, h the height of the particle at time t above the point of projection.

We have $$U^2 = 2gH, \quad h = Ut \sin \alpha - \tfrac{1}{2} g t^2.$$
$$\begin{aligned} V^2 &= U^2 \cos^2 \alpha + (U \sin \alpha - gt)^2 \\ &= U^2 + g^2 t^2 - 2gtU \sin \alpha \\ &= U^2 - 2g\,(Ut \sin \alpha - \tfrac{1}{2} g t^2) \\ &= U^2 - 2gh \\ &= 2g\,(H - h). \end{aligned}$$

Thus V is the velocity due to a fall from the directrix.

(2). *Find the time at which a particle projected with velocity U in direction α will strike a plane through the point of projection inclined at an angle β.*

We have with the same notation

$$h = Ut . \sin \alpha - \tfrac{1}{2} g t^2,$$
$$k = Ut \cos \alpha,$$
$$h = k \tan \beta.$$

Hence $$U \sin \alpha - \tfrac{1}{2} g t = U \cos \alpha \tan \beta.$$

Therefore $$t = \frac{2U}{g} (\sin \alpha - \cos \alpha \tan \beta)$$
$$= \frac{2U}{g} \frac{\sin(\alpha - \beta)}{\cos \beta}.$$

(3). *Find the angle at which a particle must be projected so as to hit a given point if the velocity of projection be given.*

Let h, k be the vertical and horizontal distances of the point from the point of projection and t the time to the point.

Then $$t = \frac{k}{U \cos \alpha},$$
$$h = Ut \sin \alpha - \tfrac{1}{2} g t^2$$
$$= k \tan \alpha - \tfrac{1}{2} g \frac{k^2}{U^2 \cos^2 \alpha}$$
$$= k \tan \alpha - \tfrac{1}{2} \frac{gk^2}{U^2} (1 + \tan^2 \alpha).$$

This gives us a quadratic equation to find α, and corresponding to the two roots we have two directions of projection.

(4). *A stone is thrown from a cliff* 112 *feet high with a velocity of* 192 *feet per second in a direction making an angle of* 30° *with the horizon; find where it strikes the ground.*

The vertical velocity is 192 sin 30 or 96 feet per second, the horizontal velocity is $96\sqrt{3}$ feet per second. When projected up with a velocity of 96 feet per second it will be at a distance of 112 feet *below* its point of projection after a time T given by

$$-112 = 96T - \tfrac{1}{2} g T^2$$
$$= 96T - 16T^2,$$
$$T^2 - 6T - 7 = 0.$$

Hence $$T = 7, \text{ or } T = -1.$$

Thus the stone will strike the ground 7 seconds after it started; we also see that the stone might have been projected from the ground with proper velocity 1 second before it started from the cliff; it would then have passed the edge of the cliff at the moment of starting with a velocity of 192 feet per second at an inclination of 30° to the horizon.

Since the horizontal velocity of the stone is $96\sqrt{3}$ feet per second, the horizontal distance from the cliff of the point at which it strikes the ground after 7 seconds will be $7 \times 96\sqrt{3}$ feet.

(5). *A man can just throw a stone 392 feet. Find the velocity with which he throws it; find also how high it will rise and determine the time of flight.*

The range is greatest when the angle of projection is 45°, and then the range is U^2/g.

Hence $$\frac{U^2}{g}=392 \text{ feet.}$$

Hence $$U^2=32\times 392,$$

whence $$U=112 \text{ feet per second.}$$

The greatest height to which it rises is

$$\tfrac{1}{2}U^2\sin^2 a/g, \text{ and } \sin^2 a=\tfrac{1}{2}.$$

Hence greatest height is $32\times 392/4\times 32$, or 98 feet.

The time of flight ($2U\sin a/g$) is $112\sqrt{2}/32$, or $7/\sqrt{2}$ seconds.

130. The Simple Pendulum. A heavy particle suspended by a fine flexible string constitutes a simple pendulum. In practice we cannot of course arrange that the string should be perfectly flexible or that the body which is suspended should be a particle. For most purposes however a spherical ball of wood or metal suspended by a piece of fine string such as a waterproofed fishing-line will serve as a simple pendulum; the ball may be conveniently from 5 to 7 cm. in diameter.

If such a pendulum be drawn aside and then let go, it commences to oscillate backwards and forwards in a vertical plane. The bob of the pendulum moves in an arc of a circle. The distance from the point of suspension to the centre of the suspended sphere is called the length of the pendulum, and we

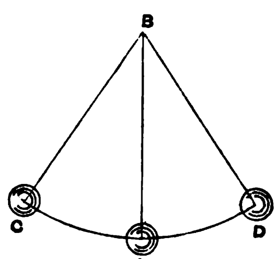

Fig. 85.

treat the motion as though the bob were a heavy particle concentrated at its centre. Such a pendulum is shewn in Fig. 85. The length of the arc AC through which the pendulum swings, measured from its lowest position, is known as the **Amplitude of the vibration.** Such a pendulum when once started loses its energy very slowly and will continue to swing for a long time; its amplitude gradually grows less but the decrease is very slow.

Now Galileo shewed that, if the amplitude of oscillation of a pendulum be not large, its time of swing is constant; thus if at starting it takes the pendulum 1 second to move from C through A to D and back to C, it will continue throughout its motion to take 1 second for each such oscillation. The uniform rate of a clock depends on this property of a pendulum. In order to verify it completely we should need to start a long heavy pendulum and count the number of oscillations in the interval between some two astronomical occurrences, which are always separated by a constant interval of time, such as the transit across the meridian of two known stars. If we find that during any such interval the pendulum makes a number of oscillations which is proportional to the length of the interval, we infer that the duration of each oscillation is a constant number of seconds. We thus arrive at the result that in a given locality the time of swing of a given pendulum is independent of the amplitude. But we can shew more than this, for we find also that the time of swing does not depend on the mass of the bob. The bob may be of any material, provided only the length of the pendulum remains unchanged, and the conditions such that we may treat the motion as that of a simple pendulum, the time of swing is the same. Newton called attention to this and made numerous experiments to verify the fact.

EXPERIMENT 27. *To shew that the time of swing of a simple pendulum is independent of the mass of the bob.*

Take a number of spheres of about the same size, but of different materials, and suspend them all side by side from some steady support, such as a horizontal bar, by strings of the same length (say 1 metre). This is most easily done by having an eye screwed into each sphere through which the string can

pass. The string is passed through a small hole in a small wooden block, it is then threaded through the eye of the bob and the end is secured to another hole in the same small block. The length of the string can then be adjusted by sliding the block up and down in the same way as the stay-ropes of a tent are tightened, and the friction will hold the block in any position. Adjust the strings of each pendulum carefully till all are of the same length, then start the pendulums swinging simultaneously. To do this, place a board against the spheres and push them all aside to the same extent. On withdrawing the board, the pendulums all start together and will continue if their lengths have been carefully adjusted to keep time for a large number of oscillations; thus the time of swing is independent of the mass of the bob.

If this experiment be continued for some time it will be found that the lighter spheres begin to lag behind. This, as Newton shewed, is due to the resistance of the air; if the experiment were performed in a vacuum no such effect would be noticed, the effect of the resistance of the air may be allowed for by observing the decrease that takes place in the amplitude of successive swings. When this is done it is found that the mass of the bob does not affect the time of swing.

131. Relation between Weight and Mass. This result affords a more accurate verification of the law that the weight of a body in a given locality is proportional to its mass than can be obtained from observations on a falling body—a pendulum is practically a falling body whose motion we can observe for a long period of time.

Consider now two of the pendulums. At any moment their velocities and accelerations are respectively the same. Their masses however are different, the force acting in each case is the same definite fraction of the weight of either pendulum, a fraction which depends on the inclination of the pendulum string to the vertical at that moment. Since the acceleration is the same the ratio of the force acting on each pendulum to the mass of the pendulum is the same for the two; hence the ratio of the weight of the pendulum to its mass is the same for all the pendulums, but this ratio measures g, the acceleration due to gravity, thus this quantity is the same for any two bodies.

132. Experiments with Pendulums.

EXPERIMENT 28. *To shew that the time of swing of a pendulum varies as the square root of its length.*

Fit up side by side three pendulums; let the length of one be l cm. (l may conveniently be about 35 cm.), that of the next 2^2l or $4l$ cm., and that of the third 3^2l or $9l$ cm. Start the first two vibrating simultaneously. Count the number of oscillations made by the shorter one while the longer one makes some 6 or 8, it will be found to be double the number made by the longer; for each oscillation of the long pendulum the short one makes two. The observation is most easily made by placing one hand so that the short pendulum at the extremity of each swing may just come up to it without contact, while the other hand is held in a similar position with regard to the long pendulum. It will then be found that in a given period the short pendulum approaches the one hand twice as often as the long pendulum approaches the other; the time of vibration of the long pendulum is twice that of the short, but the length of the long pendulum is four times that of the short, and two is the square root of four, hence in this case the times of swing are proportional to the square roots of the lengths. Now make similar observations with the first and third pendulums; it will be found in this case that the short pendulum makes three oscillations while the long one makes one, the times are as 1 to 3 while the lengths are as 1^2 to 3^2, thus the times are proportional to the square roots of the lengths. If a number of simple pendulums of different lengths be made to vibrate, the time of swing of each can be observed with a stop-watch, and the length of each measured. Make these observations for each pendulum, then form a table in which one column contains the observed times while the other contains the square roots of the lengths, it will be found that the corresponding entries in the two columns are proportional.

EXPERIMENT 29. *To verify the formula that in a simple pendulum* $\mathrm{t} = 2\pi\sqrt{\frac{\mathrm{l}}{\mathrm{g}}}$, *where* t *is the time of a complete oscillation in seconds,* l *the length of the pendulum in centimetres, and* g *the acceleration of a falling body in centimetres per second per second.*

One end of a thin string is fastened to a fixed support, the other is passed through a ring attached to a heavy ball and then fastened to a small piece of wood sliding on the string. In this way a simple pendulum of adjustable length is obtained. Make the pendulum about 100 cm. long and measure carefully the distance from the point of support to the centre of the ball. Place some mark, such as a vertical rod standing on the floor, to indicate the position of the pendulum at the lowest point. Start the pendulum swinging and determine with a watch the number of seconds taken by the ball to pass 51 times in the same direction through its lowest point; in reckoning the transits it is best to count the first transit as 0, the fifty-first will then reckon as 50, and the number of seconds which has elapsed will be the time of 50 vibrations. Divide this by 50, we have the time of an oscillation, let this be t seconds. If a stop-watch is used it should be started at the first transit and stopped at the fifty-first.

Now substitute in the formula the measured length of the pendulum l and the value of g (in England 981 cm. per second per second) and thus compute the value of the quantity $2\pi\sqrt{\frac{l}{g}}$; it will be found that this quantity is equal to t: alter the length of the pendulum, making it twice as long as before, and make another similar series of observations, it will be found that the time t is altered in the ratio of $\sqrt{2}$ to 1 or approximately 1·41 to 1 and the formula is again verified.

133. Period of Oscillation of a Pendulum. The formula verified by the preceding experiment can be deduced from the laws of motion when applied to the case of a simple pendulum: if we assume the formula true the same observations give us a means of determining g, for we can observe t and l, and then we have

$$t = 2\pi\sqrt{\frac{l}{g}}.$$

Thus
$$g = \frac{4\pi^2 l}{t^2},$$

and from this we can calculate g.

This is the method used to determine g with accuracy; in practice however a simple pendulum is not employed, a metal bar is made to oscillate about an axis perpendicular to its length; a formula can be obtained connecting together the time of swing, the dimensions of the bar and the value of g, and from this g can be found[1], the other quantities being known.

It is beyond our limits to prove the formula here. We may indicate how it is obtained in the following way.

Consider a particle falling down an inclined plane CA (Fig. 86). From C draw CD at right angles to the plane, and from A draw AD vertical to meet CD in D. Bisect AD in B, then a circle with B as centre will pass through D, C and A; let l be its radius, and let α be the angle which AC makes with the horizontal line AT which touches the circle at A. Let t_1 be the time taken to slide from C to A; the acceleration down the plane is $g \sin \alpha$, hence

$$AC = \tfrac{1}{2} g \sin \alpha t_1^2.$$

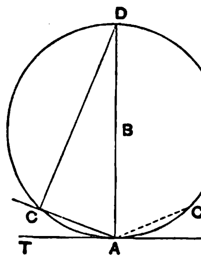

Fig. 86.

But from the figure the angle ADC is α, and $AC = AD \sin ADC = 2l \sin \alpha$.

Thus $$2l \sin \alpha = \tfrac{1}{2} g \sin \alpha t_1^2.$$

Hence $$t_1 = 2\sqrt{\frac{l}{g}}.$$

Now if we imagine a similar plane AC' on the opposite side of AD to AC, and that the particle could be started up this with the velocity it has at the bottom, it would rise to a height equal to that from which it started, and come to rest after a second interval of t_1 seconds, it would then descend and rise to C after another interval of $2t_1$ seconds; the whole time of an oscillation then would be $4t_1$, and we should have

$$\text{Time of an oscillation} = 4t_1 = 8\sqrt{\frac{l}{g}}.$$

Now if the plane is short and the angle α is small there is not much difference between the plane and the small circular arc AC, along which a particle attached to a string of length l at B would move. As a rough approximation therefore to the time of an oscillation in such a small circular arc we have the value $8\sqrt{l/g}$. Such an approximation however is only rough, for the particle starts at C along a steeper slope than the plane; its acceleration at first therefore is greater than on the plane, and

[1] See Glazebrook and Shaw, *Practical Physics*, § 20, for details of this and of the method of determining t with accuracy.

this gives it an additional velocity which more than compensates for the less acceleration of the arc near C. When allowance is made for the varying inclination of the arc, it is found that we must replace the 8 of the above formula by 2π or 6·28 approximately, and then we get

$$t = 2\pi\sqrt{\frac{l}{g}}.$$

The rough formula however illustrates the method of proof, and shews how the value $\sqrt{l/g}$ comes in. (See Section 146, Example.)

134. Value of g in different latitudes. Attention has already been called to the fact that while g is a constant for all bodies at a given point on the earth's surface, it varies from point to point. Pendulum experiments afford the best means of establishing this. If the time of oscillation of the same pendulum be observed in different latitudes it is found to vary, thus a pendulum of given length oscillates more quickly at the pole than at the equator; since the value of g is inversely proportional to the square of the time of an oscillation we can compare the values of g by comparing the square of the times of an oscillation. It was in great measure a knowledge of the fact, derived from such experiments, that g was variable which led Newton to distinguish between mass and weight.

Examples. (1). *Find the length of a pendulum which makes* 1 *complete oscillation per second at a place where the value of* g *is* 981 *cm. per sec. per sec.*

If the length of the pendulum be l cm.

then $$1 = 2\pi\sqrt{\frac{l}{981}}.$$

Hence $$l = \frac{981}{4\pi^2} = 24{\cdot}84 \text{ cm.}$$

A "seconds pendulum," as the phrase is generally employed, means one which passes through its equilibrium position once a second. Its time of swing therefore is 2 seconds and its length is four times the above.

Hence Length of a seconds pendulum under the above conditions

$$= 99{\cdot}36 \text{ cm.}$$

The actual value of g in London is 981·17, and the length of the seconds pendulum is 99·413 cm.

(2). *The value of* g *at the equator is* 978·1, *in London* 981·17, *and at the pole* 983·11; *the length of the seconds pendulum is* 99·413 *cm. in London; find its value at the pole and the equator.*

Since the time of swing is to be constant, the lengths are proportional to the values of g.

Hence

$$\text{Length at Equator} = \frac{978{\cdot}1}{981{\cdot}17} \times 99{\cdot}413 = 99{\cdot}103 \text{ cm.}$$

$$\text{Length at Pole} = \frac{983{\cdot}11}{981{\cdot}17} \times 99{\cdot}413 = 99{\cdot}610 \text{ cm.}$$

(3). *A pendulum beats seconds at London; find with the above data its time of swing at the equator.*

The length remains the same; thus the periods are inversely proportional to the square roots of the value of g.

Hence

$$\text{Period at Equator} = \sqrt{\frac{981{\cdot}17}{978{\cdot}10}}$$

$$= 1{\cdot}00156 \text{ seconds.}$$

EXAMPLES.

PROJECTILES.

1. Determine the velocity with which a stone must be projected at an angle of 45° to the horizon in order that the range may be 100 yards.

2. A stone is projected with a velocity of 50 feet per second in a direction making an angle θ with the horizon, where $\tan\theta = \frac{3}{4}$. Find the greatest height it attains.

3. If v is the vertical component of the velocity of projection of a particle, prove that the greatest height it attains above the horizontal plane through the starting-point is $\dfrac{v^2}{2g}$.

4. A stone is projected with a velocity of 60 feet per second in a direction making an angle θ with the horizon, where $\tan\theta = \frac{4}{3}$. Find its range on a horizontal plane through the starting-point, and the time of flight.

5. A cannon-ball is observed to strike the surface of the sea, to rebound, and to strike the surface again 2000 yards further on after 6 seconds. Find the horizontal velocity of the shot during the rebound and the greatest height the shot attains.

6. From the top of a vertical tower, whose height above the horizontal plane on which it stands is $\frac{13}{2}g$ feet, a heavy particle is projected with a velocity whose upward vertical and horizontal components are $6g$ and $8g$ feet per second respectively; find the time of flight and the distance from the base of the tower at which the particle will strike the ground, g being the acceleration due to gravity.

7. A particle is projected with a velocity V in a direction making an angle α with the horizon, under the action of gravity; find the highest point to which the particle will rise and the range on the horizontal plane passing through the point. If the velocity of projection be 880 feet per second, find in miles the greatest range on the plane, supposing the acceleration due to gravity to be 32 feet per second.

8. A shot is fired horizontally from the top of a tower with a velocity equal to the vertical component of its velocity when it reaches the ground; shew that it reaches the ground at a distance from the foot of the tower equal to twice the height of the tower.

9. A particle is projected with a velocity u in a direction making an angle θ with the horizontal. Find the range on the horizontal plane through the point of projection.

If the range is 300 feet, and the time of flight 5 seconds, find the velocity of projection.

10. A particle is projected with a velocity u in a direction making an angle θ with the vertical. Find the greatest height to which it will rise above the horizontal plane through the point of projection.

If the greatest height is 49 feet and the velocity at the highest point is 42 feet per second, find the velocity of projection.

11. If a particle be projected horizontally, with a given velocity u, along the surface of a smooth plane, inclined at an angle α to the horizontal, what will be the latus rectum of its path?

12. A bullet is fired from a gun at an elevation of 45°, with an initial velocity of 840 feet per second. Find the range on a horizontal plane through the point of projection.

If the bullet strikes a bird which rose vertically with uniform velocity from a point on the ground 500 yards distant from the gun at the instant when the shot was fired, find the velocity of the bird.

13. Prove that the range of a projectile on a horizontal plane through the point of projection is $2uv/g$, where u and v are the horizontal and vertical components of the velocity of projection, and g is the acceleration due to gravity.

14. A particle is projected with velocity V at an angle α with the horizon; find the height of the focus of the path.

15. A ball rolls from rest at the top of the roof of a house and finally strikes the ground at a distance from the house equal to its breadth. If the roof on both sides of the house makes an angle of 30° with the horizon, prove that the height of the house to the eaves of the roof is three times the sloping distance from the eaves to the top of the roof. [The top of the roof is in the middle of the house.]

16. Shew that if lines be drawn from a point to represent the velocities of a projectile at different instants their other extremities will lie on a vertical line.

17. A vertical line is divided into a number of equal parts A_1A_2, A_2A_3, A_3A_4 etc. Shew that if a particle be projected from O in the vertical plane through the line, OA_1, OA_2, OA_3 etc. will meet its path in points such that the times of flight from each to the next are all the same.

18. Find the range of a projectile on a horizontal plane passing through the point of projection; and prove that when the velocity of projection has a given value u there are two possible directions of projection such that the range has a given value R; provided R is less than a certain distance.

19. Prove that in this case the difference of the greatest heights attained in the two paths is $\frac{1}{2}\sqrt{\frac{u^4}{g^2} - R^2}$.

*CHAPTER X.

COLLISION.

135. Impact. We have seen already that experiment proves that when two bodies impinge there is neither loss nor gain of momentum, and it has been pointed out that in order to calculate the motion when the two impinging bodies do not adhere some further experimental result is necessary.

Newton's experiments already referred to afford the necessary information. He proved by measuring with the aid of the apparatus shewn in Fig. 51, Section 50, that when two spheres impinge directly their relative velocity after impact always bears a fixed ratio to that before impact; thus, if u, u' are the velocities of the spheres before impact and v, v' after impact, all estimated in the same direction, the relative velocities are $u-u'$ and $v-v'$ respectively. Now with Newton's apparatus the velocities can be measured and it is shewn as the result of experiments that the ratio of $v-v'$ to $u-u'$ is always the same for balls of the same two materials; it does not depend on the masses of the balls but only on the substances of which they are composed.

This constant ratio is found to be a negative fraction, that is to say if u is greater than u', so that $u-u'$ is positive, then v is less than v' or $v-v'$ is negative, the ball which is struck has the greater velocity after impact; the ratio is never greater than unity; in some cases the balls separate with the same relative velocity as that with which they impinged though the

direction of this velocity is changed; in general however the relative velocity is reduced by the impact. If we denote the ratio of $v-v'$ to $u-u'$ by $-e$, then the quantity e is never greater than unity: it is called the **Coefficient of Restitution.** For certain pairs of substances the coefficient of restitution is unity.

In general we have

$$\frac{v-v'}{u-u'}=-e.$$

We can now make use of these two results of experiment to solve some problems on impact.

PROPOSITION 35. *Two balls impinge directly; to find their velocities after impact in terms of the velocities before impact, their masses, and the coefficient of restitution.*

Let m, m' be the masses of the two balls A, B, Fig. 87,

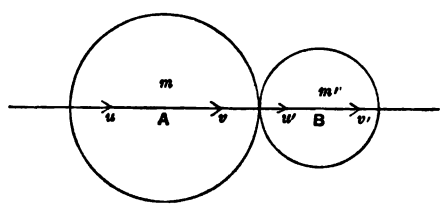

Fig. 87.

u, u' their velocities before impact, v, v' their velocities after impact, and e the coefficient of restitution.

We suppose that initially the two balls are moving in the same direction AB, the velocity (u) of A being greater than that of B.

We know that the momentum is unchanged by the impact and that the ratio of the relative velocity after impact to that before impact is $-e$.

Thus we have

$$mv+m'v'=mu+m'u',$$

$$v-v'=-e(u-u')=-eu+eu'.$$

Hence, solving these equations

$$(m+m')\,v=(m-em')\,u+m'(1+e)\,u',$$
$$(m+m')\,v'=m(1+e)\,u+(m'-em)\,u'.$$

These two equations give the values of v and v' when u and u' are known; they are simplified if e is unity, in which case the balls are said to be perfectly elastic, or if e is zero, in which case the balls adhere and move with the common velocity $(mu+m'u')/(m+m')$. In the latter case with which we have already dealt at length the balls are said to be inelastic. (See Section 58.)

PROPOSITION 36. *To find the Impulse of an Impact.*

Let the Impulse of the motion be I. Then we have

$$I=mv-mu=-(m'v'-m'u').$$

Now subtracting $(m+m')\,u$ from both sides of the equation which gives v we have

$$(m+m')(v-u)=(m-em')\,u-(m+m')\,u+m'(1+e)\,u'.$$

Hence $$v-u=\frac{m'(1+e)}{m+m'}(u'-u).$$

Therefore $$I=\frac{mm'(1+e)}{m+m'}(u'-u).$$

PROPOSITION 37. *To find the velocity of rebound after direct impact on a fixed surface.*

We may deal with this problem by supposing the mass of the second ball to be very large and its initial velocity to be zero. It will remain at rest: we must put m' infinite and u' zero in the equations and we get

$$v=-eu,\quad v'=0.$$

Or we may obtain this from Newton's second experimental result, the relative velocity before impact is u, after impact it is v, and we have $v=-eu$.

We cannot use the first experimental result, for though u' is zero, m' is very large, and we do not know what the value of $m'u'$ is.

Examples. In working examples it is much the best plan always to have recourse to the two fundamental principles and not to quote the results for v and v'. Thus

(1). *A mass of 1 kilogramme moving with a velocity of 50 cm. per second impinges directly on a mass of 10 kilogrammes at rest; the coefficient of restitution is $\frac{1}{2}$. Find the velocities after impact.*

Let the velocities after impact be v and v' respectively in centimetres per second. The momentum before impact is 50000 gramme-centimetre units, and the relative velocity 50 cm. per sec.

Hence
$$1000v + 10000v' = 50000,$$
$$v - v' = -\tfrac{1}{2}50 = -25.$$

Hence
$$10v' + v = 50,$$
$$v' - v = 25,$$
$$11v' = 75,$$
$$v' = \tfrac{75}{11} = 6\tfrac{9}{11} \text{ cm. per sec.},$$
$$v = -(25 - 6\tfrac{9}{11}) = -18\tfrac{2}{11} \text{ cm. per sec.}$$

Thus the 1 kilogramme ball returns with a velocity of $18\frac{2}{11}$ cm. per sec., the 10 kilogramme ball moves forward with a velocity of $6\frac{9}{11}$ cm. per sec.

(2). *A ball whose mass is 1 lb. moving with a velocity of 10 feet per second impinges directly on another ball moving in the opposite direction with a velocity of 5 feet per second. The first ball is observed after impact to continue to move on with a velocity of 5 feet per second, and the coefficient of restitution is $\frac{2}{3}$. Find the mass and the velocity after impact of the second ball.*

Let the mass be m' lb. and the velocity estimated in the direction in which the 1 lb. ball moves v' ft. per sec. The momentum before impact in this direction is $10 - 5m'$.

The relative velocity before impact is $10 + 5$ or 15.

Hence
$$5 + m'v' = 10 - 5m',$$
$$5 - v' = -\tfrac{2}{3}15 = -10.$$
$$\therefore\ v' = 15.$$

Hence
$$5 + 15m' = 10 - 5m',$$

or
$$20m' = 5,$$
$$m' = \tfrac{1}{4}.$$

Thus the mass of the second ball is $\frac{1}{4}$ of a lb. and it moves after impact in the direction opposite to that of its initial motion with a velocity of 15 feet per second.

(3). *A ball drops from a height of 25 feet on to a horizontal surface and rebounds, the coefficient of restitution is $\frac{3}{4}$; find how high the ball will rise after striking the surface three times.*

Let the velocity with which it strikes the surface be u ft. per sec. Then since the velocity is acquired by falling through 25 feet

$$u = \sqrt{2g \cdot 25} = \sqrt{64 \times 25} = 40 \text{ feet per sec.}$$

After 1 rebound this becomes $\frac{3}{4}$ of 40, or 30 feet per second.

The ball rises and then falls again, striking the ground with a velocity of 30 feet per second. After this second impact the velocity becomes $\frac{3}{4}$ of 30, and after the third impact it becomes

$$\tfrac{3}{4} \times \tfrac{3}{4} \times 30 \text{ or } \tfrac{9}{8} \times 15.$$

The height to which the ball will rise then is

$$\left(\frac{9 \times 15}{8}\right)^2 \Big/ 2g.$$

Thus the height required is equal to

$$\left(\frac{135}{64}\right)^2, \text{ or about } 4{\cdot}448 \text{ feet.}$$

136. Energy and Impact.

PROPOSITION 38. *To find the work done when two bodies impinge directly.*

The kinetic energy of each ball is changed by the impact; thus work is done, and the work done on either ball is equal to its gain of kinetic energy.

Hence the work done on the ball A

$$\begin{aligned} &= \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 \\ &= \tfrac{1}{2}m(v-u)(v+u) \\ &= \tfrac{1}{2}I(v+u), \end{aligned}$$

if I is the impulse or whole change of momentum.

The work done on the ball B

$$\begin{aligned} &= \tfrac{1}{2}m'v'^2 - \tfrac{1}{2}m'u'^2 \\ &= \tfrac{1}{2}m'(v'-u')(v'+u') \\ &= -\tfrac{1}{2}I(v'+u'), \end{aligned}$$

for $m'(v'-u') = -m(v-u) = -I$.

PROPOSITION 39. *To find the whole change of kinetic energy on Impact.*

We have just seen that the ball A gains an amount $\frac{1}{2}I(v+u)$ of energy, while the ball B loses an amount

$$\tfrac{1}{2}I(v'+u').$$

Hence the total loss of kinetic energy is

$$\tfrac{1}{2}I(v'+u'-v-u).$$

Now $$v-v'=-e(u-u').$$

Thus the loss of kinetic energy is

$$\tfrac{1}{2}I(u'-u)(1-e).$$

On substituting the value of I from Prop. 38, we find for the loss of kinetic energy

$$\frac{1}{2}\frac{mm'}{m+m'}(1-e^2)(u'-u)^2.$$

Now $(u'-u)^2$ being a square is always positive and e^2 is not greater than unity, hence $1-e^2$ is positive unless $e=1$, when it is zero. Thus the loss of kinetic energy on impact is always positive unless the coefficient of restitution is unity, when there is no loss. In this case kinetic energy is transferred from one ball to the other but its amount is unchanged. In general however kinetic energy disappears. Joule's experiments, already referred to, lead us to believe that the total energy is unchanged, the energy apparently lost in the kinetic form is transformed into heat, the balls are raised in temperature, and the heat needed for this is measured by the loss of kinetic energy.

137. Oblique Impact. In the cases of impact which have been considered it has been supposed that the two balls were moving either in the same or in exactly opposite directions at the point of impact. This is not always the case: consider the two balls A, B, Fig. 88, let their directions of motion make angles a, a' before impact with the line joining their centres, which is known as the line of impact, and β, β' with the same line after impact, their velocities and masses being as in the previous sections. In considering the problem we have to deal

with the motion along the line AB and also with that at right angles to it. Now the whole momentum is unchanged, and

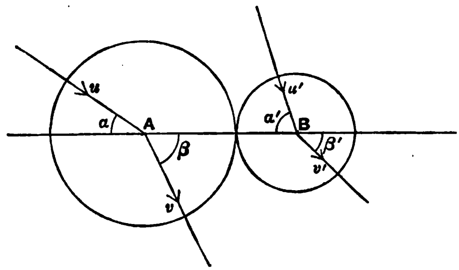

Fig. 88.

this statement is true for the components of the momentum along and perpendicular to the line of impact.

If the balls be smooth there is no force between them at right angles to the line of impact. Thus the velocity of each ball in this direction remains unchanged.

Newton's experimental result as to the relative velocity before and after impact applies to the velocities along the line of impact.

PROPOSITION 40. *To determine the motion after impact of two balls which impinge obliquely.*

The first principle above stated gives us the following results:

$$mv\cos\beta + m'v'\cos\beta' = mu\cos\alpha + m'u'\cos\alpha' \quad\text{......(i)},$$

$$mv\sin\beta + m'v'\sin\beta' = mu\sin\alpha + m'u'\sin\alpha' \quad\text{......(ii)}.$$

From the second we have

$$v\sin\beta = u\sin\alpha \quad\text{....................(iii)},$$

$$v'\sin\beta' = u'\sin\alpha' \quad\text{.................(iv)},$$

while from the third we find that

$$v\cos\beta - v'\cos\beta' = -e\,(u\cos\alpha - u'\cos\alpha') \quad\text{......(v)}.$$

Of these five equations (ii) is included in (iii) and (iv). Hence we have four independent equations (i), (iii), (iv) and (v) from which we can find v, v', β and β'.

We may shew as above that kinetic energy is lost by the impact and that the amount so lost is

$$\frac{1}{2}\frac{mm'}{m+m'}(1-e^2)(u\cos\alpha-u'\cos\alpha')^2.$$

If one of the bodies be fixed we proceed as follows:

Example. *A billiard-ball moving with velocity* u *strikes the cushion at an angle* α, *the coefficient of restitution being* e; *find the direction and velocity of rebound.*

Let v be the velocity of rebound and let the direction of motion after impact make an angle β with the normal to the cushion at the point of impact.

Then the velocity along the cushion is unchanged, that perpendicular to the cushion is reversed and reduced in the ratio e to 1.

Hence if the velocities be estimated as in Fig. 89, we have

$$v\sin\beta=u\sin\alpha,$$
$$v\cos\beta=eu\cos\alpha$$

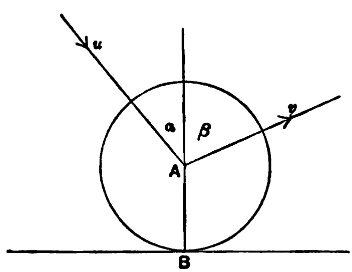

Fig. 89.

Thus
$$v^2=u^2(\sin^2\alpha+e^2\cos^2\alpha),$$
$$\cot\beta=e\cot\alpha.$$

If the coefficient of restitution be unity, we have

$$e=1, \text{ then } v=u,\ \beta=\alpha.$$

The ball rebounds with its velocity unchanged in amount, and its direction of motion inclined to the cushion at the same angle as before, but on the opposite side of the normal.

138. Action during Impact. The term **Action** has been used in our discussion of the third law either for the rate

at which momentum is transferred, or the rate at which work is done. Now in a case of impact the momentum of each ball is changed by a finite amount in a very brief period, the rate of change of momentum is very great, too great for observation; we do not deal with the rate of change of momentum but with the whole change which occurs during the impact. This whole change is what is meant by the Impulse of the motion, so that *when we speak of the* Action *taking place between two bodies at impact we refer to the whole amount of momentum which is transferred.*

139. Moment of greatest compression. It is convenient in some cases to divide the whole change of momentum into two parts. Consider what takes place at the point of impact: the bodies are deformed, the velocity of the one A is being reduced, that of the other B is being increased, there will be a moment during the very brief interval in which the two are in contact at which the deformation of each ball has reached its maximum amount and the two balls are moving together with the same velocity. Let us then divide the duration of the impact into two parts, the first lasting up to this instant of greatest compression, the second, during which the balls are again separating, from this instant up to the time at which contact ceases. Let I_1 be the momentum transferred in the first part, I_2 in the second, then we have

$$I = I_1 + I_2.$$

PROPOSITION 41. *To shew that the impulse after the moment of greatest compression bears a constant ratio to that before, and that this ratio is the coefficient of restitution.*

By hypothesis the two balls have a common velocity at the instant at which an amount I_1 of momentum has been transferred. Let this velocity be V; then

$$m(V-u) = I_1 = -m'(V-u').$$

Hence
$$V = \frac{mu + m'u'}{m + m'}.$$

Thus
$$I_1 = \frac{mm'}{m + m'}(u' - u).$$

But $$I = \frac{mm'(1+e)}{m+m'}(u'-u).$$

Hence $$I = (1+e)I_1.$$

But $$I = I_1 + I_2.$$

Therefore $$I_2 = eI_1.$$

Hence the impulse during the second period of the impact bears a constant ratio to that during the first part, and this ratio is measured by the coefficient of restitution.

Thus when a ball impinges directly on a flat surface and rebounds, during the first part of the impact all its momentum is transferred to the surface and the ball is reduced to rest, during the second period an amount of momentum e times as great as that which it has lost is acquired by the ball in the opposite direction, it rebounds therefore with a velocity e times as great as that with which it struck the surface.

Example. *A ball whose mass is 1 lb. moving with a velocity of 10 feet per second overtakes another ball whose mass is $\frac{1}{4}$ lb. moving in the same direction with a velocity of 5 feet per second. The coefficient of restitution is $\frac{3}{4}$; find the impulse up to the moment of greatest compression and the whole impulse.*

Let V be the common velocity at the moment of greatest compression. Then since the momentum is unchanged

$$(1+\tfrac{1}{4})V = 1 \times 10 + \tfrac{1}{4} \times 5,$$

$$\frac{5V}{4} = \tfrac{45}{4},$$

$$V = 9 \text{ feet per second.}$$

Thus the Impulse up to this time is $1(9-10)$ or -1.

The ball loses 1 lb.-foot unit of momentum.

The total impulse therefore is $1(1+\frac{3}{4})$ or $\frac{7}{4}$ lb.-foot units of momentum. These are lost by the first ball, gained by the second. Hence if v' be its final velocity we have

$$\tfrac{1}{4}v' = \tfrac{7}{4} + \tfrac{5}{4}.$$

Thus after impact the smaller ball has a velocity of 12 feet per second.

EXAMPLES.

IMPACT.

1. An inelastic ball impinges directly on another of half its mass at rest, find the new velocity of the two in terms of that of the impinging ball.

2. The velocities of two balls before impact are 10 and 6 feet per second respectively, after impact they are 5 and 8 feet per second respectively; compare the masses of the two balls and find the coefficient of restitution.

3. Two bodies of unequal mass moving in opposite directions with equal momenta impinge directly. Shew that their momenta are equal after impact.

4. A body whose mass is 8 lb. impinges directly on one whose mass is 1 lb., the coefficient of restitution is $\frac{1}{3}$. After impact the momenta of the two balls is the same and the smaller has a velocity of 15 feet per second; find the original velocities of the two.

5. Find the condition that two balls may interchange velocities on direct impact.

6. A body is dropped from a height of 64 feet on to a horizontal floor. If the coefficient of restitution be $\frac{1}{2}$, find the height to which the body rises after 3 rebounds.

7. A ball strikes a cushion at an angle of 45°. If the coefficient of restitution be $\frac{2}{3}$, find the angle of rebound.

8. A ball impinges with a velocity u on an equal ball at rest, the direction of motion making an angle of 30° with the line of the centres; determine the motion of the two balls afterwards, the coefficient of restitution being unity.

9. A ball impinges obliquely on an equal ball at rest, find the direction of impact if the two move afterwards with equal velocities, the coefficient of restitution being unity.

10. Two balls of masses m, m', whose coefficient of elasticity is e, impinge directly on each other with velocities u, v respectively ($u>v$). Shew that the impulse between the balls is

$$\frac{mm'}{m+m'}(u-v)(1+e).$$

11. Of three equal balls A, B, C, placed in this order in one straight line, A moves with a given initial velocity towards B, while B and C are at rest. Find B's velocity after it has struck C, assuming the coefficients of elasticity to be the same for the two pairs of balls A, B and B, C.

12. Find the velocities after impact of two smooth spheres which impinge directly, in terms of their masses, their velocities before impact, and the coefficient of restitution.

Shew that the velocity of their centre of inertia is unaltered by the impact, and that the velocity of each body relative to the centre of inertia is reversed in direction and diminished in the ratio of $1 : e$, where e is the coefficient of restitution.

13. Two smooth spherical masses m and m' moving with given velocities u and u' in the same direction collide. Shew that the loss of kinetic energy due to the collision is

$$\frac{mm'}{2(m+m')}(u-u')^2(1-e^2),$$

where e is the coefficient of restitution.

*CHAPTER XI.

MOTION IN A CIRCLE. MISCELLANEOUS.

140. The Hodograph. The velocity of a moving particle may be represented at any moment by a straight line drawn from some fixed point, the length of the line represents the magnitude while its direction represents the direction of the velocity. Thus if the particle move with constant velocity the straight line is fixed in magnitude and direction. If the particle move with uniform acceleration in a straight line the direction of the line representing the velocity is fixed; its length however increases uniformly with the time. Thus if OQ represent the velocity at any moment, and if OQ_1

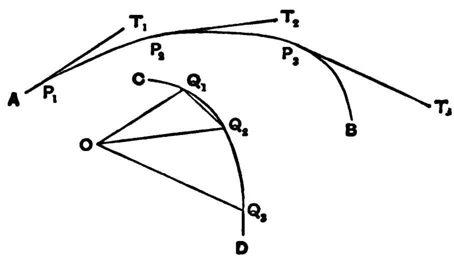

Fig. 90.

represent it after an interval of 1 second, then QQ_1 is the increase of velocity in 1 second, and is therefore equal to the acceleration.

Consider now a particle moving in a curve AB, Fig. 90, let $P_1, P_2 \ldots$ be its positions at different times and draw P_1T_1, $P_2T_2 \ldots$

to represent the velocities V_1, V_2... of the particle in those positions. From a fixed point O draw OQ_1 equal and parallel to V_1, OQ_2 equal and parallel to V_2, and so on for the other points P_3 etc.

We thus get a second series of points Q_1, Q_2, Q_3... which have the property that the lines drawn to these points from the fixed point O represent the velocities of the particle in the corresponding positions P_1, P_2 etc.

If a similar construction be made for all points on the curve AB we shall get a second curve CD, which has the property that the lines drawn to it from O represent in direction and magnitude the velocity of the particle at the corresponding points of AB.

This second curve CD is called the hodograph of AB.

141. The Hodograph and the Measurement of Acceleration. Again let Q_1, Q_2 in Fig. 90 be two points on the hodograph, P_1, P_2 the corresponding points on the path; in moving from P_1 to P_2 the velocity changes from OQ_1 to OQ_2, join Q_1Q_2; then the change in velocity is represented in direction and magnitude by Q_1Q_2, for Q_1Q_2 represents a velocity which when compounded with OQ_1 will give OQ_2.

PROPOSITION 42. *To shew that in the case of uniform acceleration the hodograph is a straight line.*

If P_1, P_2, Fig. 91, are two positions which the particle occupies after an interval of 1 second, then Q_1Q_2 is the change in velocity in 1 second; if we know that the acceleration is constant then it is measured by the change in velocity in 1 second. Hence in this case the line Q_1Q_2 represents the acceleration. Now let P_3 be the position of the particle after a further interval of 1 second and Q_3 the corresponding point on the hodograph. Then in the same way Q_2Q_3 represents the acceleration, but since the acceleration is

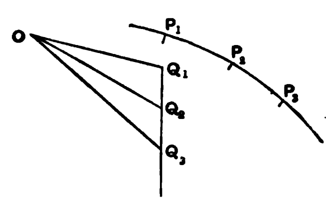

Fig. 91.

constant in direction and magnitude Q_2Q_3 must be equal to and in the same straight line as Q_1Q_2. Thus the hodograph $Q_1Q_2Q_3 \dots$ must in this case be a straight line.

Hence if a particle move with uniform acceleration the hodograph is a straight line, and the arc of the hodograph traced out in 1 second represents the acceleration.

This proposition can be generalised thus.

PROPOSITION 43. *If* P *be a particle describing any curve and* Q *the point on the hodograph which corresponds to* P, *then the velocity of* Q *in the hodograph measures the acceleration of* P.

For suppose that after a short time τ, P, Fig. 92, has moved to P', and Q in consequence to Q'.

Then the velocity of Q is given by the ratio QQ'/τ, when τ is made very small. For QQ' is the space traversed by Q in time τ, and the ratio of the space traversed to the small time of traversing it measures the velocity of Q.

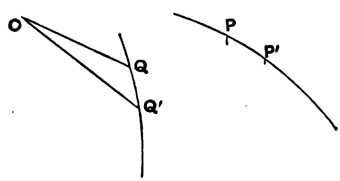

Fig. 92.

But when τ is very small, we may treat QQ' as a straight line: moreover we may consider the acceleration of P as constant for the interval τ, if that interval be made sufficiently small.

Now OQ is the original velocity of P and OQ' is its velocity after the interval τ. Hence QQ' is the change in velocity during the interval and the ratio of this change to the interval in which it occurs measures the acceleration of P.

Thus the ratio QQ'/τ measures the acceleration of P as well as the velocity of Q. Hence the acceleration of P is equal to the velocity of Q.

Whenever then we know the hodograph of a path and the velocity with which it is described, we can find the acceleration in the original path.

Thus when the hodograph is a straight line described with uniform velocity, the acceleration is constant both in direction and magnitude;

when the hodograph is a straight line but the velocity in it is not uniform the acceleration is constant in direction but variable in amount.

We proceed to give some other examples.

PROPOSITION 44. *A particle describes a circle with uniform speed* V; *to find the hodograph and the acceleration.*

Let the path of the particle be a circle P_1P_2 with centre C, Fig. 93. Let P_1 be the position of the particle in the original path, P_2 its position after one second.

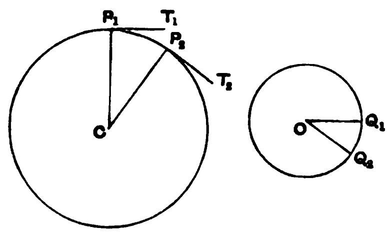

Fig. 93.

At P_1, P_2 draw P_1T_1, P_2T_2 to represent the velocity: since the speed is constant $P_1T_1 = P_2T_2 = V$.

Again, since the path is described with uniform speed and P_1P_2 is the distance traversed in a unit of time we have P_1P_2 equal to V.

Now let OQ_1 equal and parallel to P_1T_1 represent in direction and magnitude the velocity at P_1, and let OQ_2 represent it at P_2.

Then OQ_2 is parallel to P_2T_2. Hence OQ_1 and OQ_2 are respectively perpendicular to CP_1 and CP_2.

Hence the angle Q_1OQ_2 is equal to the angle P_1CP_2.

Since the speed is constant the length of the radius vector from O is constant. Thus the hodograph is a circle and this circle is described by the point Q with uniform speed.

Now Q_1Q_2 represents the space traversed in 1 second by the point in the hodograph when moving with uniform velocity. It is therefore equal to the velocity in the hodograph. Thus it

represents the acceleration completely. Also since the velocity at Q_1 is perpendicular to OQ_1, the acceleration at P_1 is perpendicular to P_1T_1, that is, it is along P_1C. Hence the acceleration at P_1 is represented in amount by Q_1Q_2, the arc of the hodograph described in one second, and is directed along P_1C.

Hence if a be the acceleration we have

$$Q_1Q_2 = a,$$
$$OQ_1 = V.$$

Let r be the radius of the circle P_1P_2.

Then since the angles Q_1OQ_2 and P_1CP_2 are equal their circular measures are the same.

Hence $$\frac{Q_1Q_2}{Q_1O} = \frac{P_1P_2}{P_1C},$$

or $$\frac{a}{V} = \frac{V}{r}.$$

Thus $$a = \frac{V^2}{r}.$$

Hence when a particle moves with uniform speed V in a circle of radius r, it has at each point an acceleration directed to the centre of the circle and equal in amount to $\frac{V^2}{r}$.

Thus the force towards the centre is

$$\frac{mV^2}{r},$$

if m is the mass of the particle.

By expressing these results in terms of the angular velocity of the particle we can put them in slightly different form, for if Ω be the angular velocity we have (Section 38)

$$V = \Omega r.$$

Hence $$a = \frac{V^2}{r} = \Omega^2 r,$$

$$F = \frac{mV^2}{r} = m\Omega^2 r.$$

Thus when we observe a body moving in a circle with uniform speed we know that it has the acceleration given above.

If a stone is tied to a string and swung round in a horizontal circle, force toward the centre is exerted by the string; this force measures the tension of the string; if we call it T then

$$T=\frac{mV^2}{r}=m\Omega^2 r.$$

The string breaks when the angular velocity is such as to make this tension greater than it can bear.

Example. *A string* 1 *metre long can support a body whose mass is* 10 *kilogrammes. A mass of* 100 *grammes is tied to one end and whirled in a horizontal circle making one revolution per second; find the tension of the string; find also the greatest number of revolutions per second which can be given to the mass without breaking the string, and calculate the kinetic energy of the mass when moving with this greatest possible speed.*

When the mass makes 1 revolution per second, an angle whose circular measure is 2π is traced out in each second; thus the angular velocity is 2π.

Hence since $r=100$ cm. the acceleration is

$$4\pi^2\, 100 \text{ cm. per sec. per sec.},$$

and the force is

$$4\pi^2\, 100\times 100,$$

or $$394880 \text{ dynes approximately.}$$

When the mass makes n revolutions per second, the angular velocity is $2\pi n$, and the tension

$$4\pi^2 n^2 10^4 \text{ dynes},$$

or approximately $$394880n^2 \text{ dynes.}$$

Now the breaking tension is the weight of 10 kilogrammes or

$$10\times 981\times 1000 \text{ dynes.}$$

Hence to find the maximum number of revolutions per second we have $$394880n^2=981\times 10^4.$$

Thus $$n^2=\frac{981000}{39488}=24{\cdot}85.$$

Hence $n=4{\cdot}98$ or very nearly 5.

Hence the string will break before the mass attains a speed of 5 revolutions per second.

The kinetic energy is $\frac{1}{2}mV^2$ ergs.

The value of this is

$$\tfrac{1}{2}\times 4\pi^2\times 100n^2\times 100^2 \text{ ergs},$$

or $$490{\cdot}8\times 10^6 \text{ ergs.}$$

142. Motion in a circle. Consider a body such as a marble in a horizontal tube along which it can move freely: if the tube be set spinning about a vertical axis, the marble will be shot out at one end; now suppose the marble attached by a spiral spring or piece of elastic to some point in the tube, the spring will be stretched until the tension it can exert on the marble and the impressed force mv^2/r on the marble are equal.

Suppose now that the marble is attached to two points in the tube as shewn at *AB*, Fig. 94, by two spiral springs, the

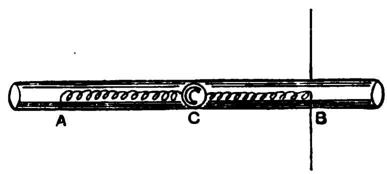

Fig. 94.

marble being between *A* and *B*. Both these springs are stretched until each is subject to the same tension. The one spring *AC* may represent a spiral spring balance by which the marble is suspended, the tension in the other spring will then stand for the force of attraction between the earth and the marble, the weight of the marble; this weight is measured by the extension of the balance *AC*. Now set the tube rotating about a vertical axis through *B*, the marble will move towards *A*, the spring *AC* will be less stretched than before, the weight of the marble as measured by the extension of this spring will appear less. If when the tube were at rest the spring *AC* were cut, the marble would move towards *B* with an acceleration depending on the extension of the spring *BC*; if when the tube is rotating the spring *AC* be cut, the marble unless the rotation be too great will move towards *B*, but its acceleration will be less than it was before by the amount $\Omega^2 r$, where Ω is the angular velocity and r the distance from the axis of rotation.

The acceleration with which the marble moves towards *B* stands for the acceleration with which a body falls under

gravity; this, other things being equal, is greater when the tube is at rest than it is when it is moving round an axis.

143. Consequences of the Earth's Rotation. Now let us apply this to the Earth. A particle on the Earth is describing a circle about the axis of the Earth; near the pole this circle is small, near the equator it is large, the angular velocity, however, is the same in all cases. We may illustrate this in a rough manner by supposing that in order to represent a particle near the pole with the tube, the axis of rotation passes near to the marble, while to represent a particle near the equator the axis of rotation is far from the marble.

In the first case the marble will be very slightly disturbed by the rotation, its weight as measured by the spring AC will be nearly the same as it was when the tube was at rest; in the second case the marble may be considerably displaced, its weight is appreciably diminished; thus we see how in consequence of the rotation of the Earth the acceleration of a falling body is less near the equator than near the pole.

144. Circular Motion. It follows then that in order that a body may move in a circle with uniform speed, its connexion with some other body must be such as to be consistent with its having an acceleration v^2/r towards the centre of the circle, if this is not the case the body will not move in the circle.

When the marble is free in the tube, it cannot, under the normal pressure of the walls, acquire an acceleration towards the centre and is shot out of the rotating tube; when it is connected to the spring, the spring is stretched until this acceleration is acquired and then the circular motion continues.

Many other examples might be given. Thus consider a body C suspended by a vertical rod BC from a horizontal arm AB: let there be a joint at B which will allow the body to move in the vertical plane ABC. Cause the whole to rotate about a vertical axis through A, the body C will rise, the rod BC' no longer remaining vertical. For if the rod be vertical the body is in equilibrium under its weight and the tension of the rod, it cannot then have an acceleration towards the centre of

rotation on the axis through A. When the body rises the tension is no longer vertical and we can resolve it into two

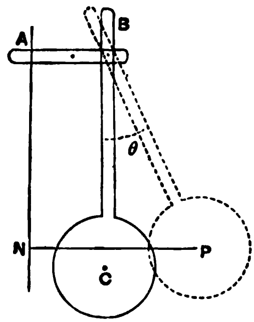

Fig. 95.

components, the one horizontal, the other vertical: since when it is rotating steadily the body has no vertical acceleration, the vertical component of the tension balances the weight: under the horizontal component the body acquires the acceleration $\Omega^2 r$ necessary for its circular motion.

We can find the position of the body thus: let P be its position and let θ be the angle which the rod PB makes with the vertical. Let $AB = a$, and $BP = b$. Draw PN on the vertical through A. Let m be the mass of the body, T the tension of the rod.

Then $$r = PN = a + b \sin \theta.$$

Hence resolving horizontally

$$T \sin \theta = m\Omega^2 r = m\Omega^2 (a + b \sin \theta),$$

$$T \cos \theta = mg.$$

Therefore

$$\tan \theta = \frac{(a + b \sin \theta)\,\Omega^2}{g}.$$

From this equation we can find θ, and then θ being known the tension is given by the equation

$$T = mg \sec \theta.$$

An arrangement of this description is made use of in the ball governors attached to some forms of steam-engine. Mechanism for opening and closing the steam ports so as to vary the supply of steam is connected with the ball. When the engine runs too fast the ball rises and the steam port is closed, when the speed is too slow the port is opened, for the ball falls. Watt's governor is shewn in Fig. 96. In this form there are two balls, and the value of a is zero.

Hence $T = \Omega^2 mb = mg \sec\theta$.

Therefore for the equilibrium position

$$\cos\theta = g/\Omega^2 b.$$

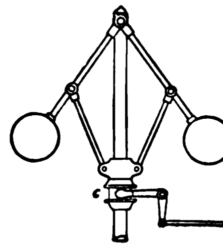

Fig. 96.

145. The shape of the Earth. When dealing with the variation of g attention was called to the fact that the Earth was not spherical but was flattened at the poles; this again is a consequence of the rotation. This may be shewn by spinning, about a vertical axis, a circular hoop, such as that shewn in Fig. 97, of thin brass or some other elastic material.

The hoop is fixed at the bottom to the axis, at the top the axis passes through a collar attached to the hoop in which it can slip freely.

Fig. 97.

When the hoop is rotating with uniform speed there must be a force on each particle of mass m at a distance r from the axis equal to $m\omega^2 r$.

Unless this force is exerted the particle cannot move uniformly in a circle. When the motion is first started the action between the various parts of the hoop is not such as to give rise to this force. At first therefore each particle does not move uniformly in a circle, it also moves outwards from the axis; by this motion the hoop is bent from its circular form, becoming flattened at the top and bottom, and this bending continues until the acceleration acquired by the particles under the forces to which the bending gives rise is that requisite to give uniform motion in a circle.

The same may be shewn by floating a spherical bubble of

oil in a mixture of alcohol and water. Such a bubble can be set into rotation, and when this is done it becomes flattened at the points in which its surface is met by the axis. Now the material of which the Earth is composed resembles in some respects the brass hoop or the oil of the bubble, the Earth is rotating round its axis and hence its shape is not spherical but oval, flattened at the poles.

146. Simple harmonic motion. Let ACB, Fig. 98, be a diameter of a circle centre C, and suppose a particle P is moving round this circle with uniform angular velocity Ω. Let it start initially from the point A and let it be at the point P after t seconds.

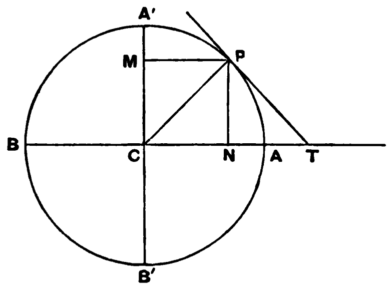

Fig. 98.

Let $A'CB'$ be the diameter perpendicular to ACB and draw PN, PM perpendicular to AB and $A'B'$ respectively.

Then in t seconds the radius CP traces out the angle PCA.

Hence $\quad PCA = \Omega t.$

Let $\quad CP = a.$

Then $\quad CN = CP \cos PCA = a \cos \Omega t,$

$\quad CM = CP \sin PCA = a \sin \Omega t.$

Now as P moves uniformly round from A, the point N starts from A and moves along AC towards C; when P is at A', N has reached C; as P moves on to B, N moves towards B, coinciding with P at B; as P moves back along BB' to A, N moves back to A. Thus N has a vibratory motion backwards and forwards along AB and its distance x from C is always given by the equation $\quad x = a \cos \Omega t.$

The motion of N is said to be **simple harmonic** *motion.*

The motion of M is also simple harmonic.

PROPOSITION 45. *To find the velocity of a particle moving with simple harmonic motion.*

The velocity of P, Fig. 98, may be resolved into two components, one along AC and the other along CA', the former of these gives the velocity of N.

Now the velocity of P is V at right angles to CP. Draw PT at right angles to CP meeting CA produced in T; then a velocity V along TP has for its component along AC a velocity

$$V \cos PTC.$$

But CPT is a right angle.

Hence $\qquad \cos PTC = \sin PCA = \sin \Omega t.$

Thus the velocity of N is $V \sin \Omega t$.

Also $V = a\Omega$; we have therefore the result that

When the distance of a particle, oscillating about a fixed point, from that fixed point is given by

$$x = a \cos \Omega t,$$

then the velocity of the particle toward that point is given by

$$v = a\Omega \sin \Omega t.$$

PROPOSITION 46. *To find the acceleration of a particle moving with simple harmonic motion.*

The acceleration of P is $\Omega^2 a$ towards C. This can be resolved into $\Omega^2 a \cos PCA$ along NC and $\Omega^2 a \sin PCA$ along MC. The first of these is the acceleration of N.

Hence the acceleration of N is given by

$$\Omega^2 a \cos PCN \text{ or } \Omega^2 a \cos \Omega t.$$

But $\qquad CN = a \cos PCN.$

Thus the acceleration required is

$$\Omega^2 CN \text{ or } \Omega^2 x.$$

We have thus the result that when the acceleration of a particle moving in a straight line is always directed to a fixed point in the line and is proportional to the distance of the particle from that point, then the motion is simple harmonic.

Moreover if the acceleration at any distance x is equal to μx, then the value of x in terms of the time is given by

$$x = a \cos \sqrt{\mu}\, t,$$

where a gives the distance of the particle from the fixed point initially.

This follows from the result that if the distance is

$$x = a\cos\Omega t,$$

then the acceleration is $\Omega^2 x$; hence if the acceleration is μx we have

$$\Omega^2 = \mu \text{ and } x = a\cos\sqrt{\mu}\,t.$$

Moreover if T be the time of a complete revolution, then in T seconds the radius traces out four right angles: hence

$$\Omega T = 2\pi.$$

Thus if the acceleration at distance x be μx,

$$\text{the Periodic time} = \frac{2\pi}{\Omega} = \frac{2\pi}{\sqrt{\mu}}.$$

Thus if a particle move in a straight line under an acceleration μx towards a fixed point the motion is a simple harmonic vibration; the distance x of the particle from the fixed point at any time t is given by $x = a\cos\sqrt{\mu}\,t$, its velocity towards the fixed point is $v = a\mu\sin\sqrt{\mu}\,t$, and T the time of a complete oscillation is found from the equation

$$T = \frac{2\pi}{\sqrt{\mu}}.$$

It is not necessary for the motion to take place in a straight line. We may suppose the particle to be a ring oscillating backwards and forwards on a smooth straight wire, the acceleration being directed to a fixed point on the wire. Then the wire may be bent into any shape without altering the motion, provided that the acceleration at each point of the wire remain of the same value as before. This will be secured if the acceleration be proportional to the distance of the particle from the fixed point measured along the wire, the particle still having the same harmonic motion as before. Thus a particle moving on a smooth curve with an acceleration directed along the curve to some fixed point on the curve, and equal to $\mu \times$ {distance measured along the curve of the particle from the fixed point}, has simple harmonic motion about the point, the period of the motion being $2\pi/\sqrt{\mu}$.

Example. *A particle suspended by a string of length* l *oscillates in a vertical circle of radius* l; *find the time of a complete oscillation.*

Let the string CP, Fig. 99, make an angle θ with the vertical line CA.

Let A be the equilibrium position, and let the length of the arc AP be s.

The acceleration of the particle along the curve is $g \sin \theta$.

Now if θ be small $\sin \theta = \theta = s/l$.

Hence acceleration of particle $= \dfrac{g}{l} s$.

Thus the motion is simple harmonic, and

$$s = s_0 \cos \sqrt{\frac{g}{l}}\, t,$$

while the periodic time is $2\pi \sqrt{\dfrac{l}{g}}$.

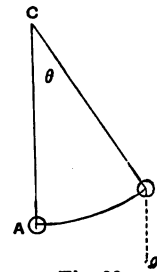

Fig. 99.

This example gives us the case of a simple pendulum, and we have thus proved the formula for the time of swing which was verified by experiment in Experiment 29. We see moreover that this formula is only approximate, for the motion is not simple harmonic unless θ is so small that we may treat θ and $\sin \theta$ as equal. For other applications of this method of dealing with problems in simple harmonic motion, see Maxwell, *Matter and Motion*, Article cxvi. and following.

EXAMPLES.

CIRCULAR AND HARMONIC MOTION. THE PENDULUM.

1. Explain what is meant by simple harmonic motion.

Describe some method of causing a particle to move with simple harmonic motion, and determine the energy of such a particle.

2. A mass of one pound is attached to a string three feet long and whirled in a horizontal circle. If the string can just carry a weight of ten pounds, find how many revolutions per second the mass can make without breaking the string.

3. A clock whose pendulum ought to beat seconds gains at the rate of 10 minutes a week. What alteration must be made in the length of the pendulum to correct its error? The length of the simple seconds pendulum is about 39·12 inches.

4. Define the terms Acceleration, Momentum, Kinetic Energy, and determine their values in the case of a simple pendulum which has been drawn aside through a known angle and then released.

5. A pendulum consisting of a bob weighing 1 kilogramme at the end of a string 1 metre long is drawn aside until the bob is 25 cm. from the vertical through the point of support, and is held in this position by a horizontal string. Find the forces on the bob (1) when in this position, (2) just after the horizontal string is cut, (3) as the bob swings through its lowest position.

6. Shew that there is a force acting on a body which moves with uniform velocity in a circle.

7. A pendulum which beats seconds at a place A gains 10 seconds a day when taken to another place B; compare the intensity of gravity at A and B.

8. Describe an arrangement to exhibit the combination of two simple harmonic vibrations in planes at right angles. Point out how such an arrangement may be used to illustrate or verify the law of the addition of vector quantities.

9. Define a simple harmonic vibration and shew that such a vibration is executed when a particle moves under the action of a force which varies as the distance of the particle from its position of equilibrium.

10. A bullet fired from a gun hits the bob of a heavy simple pendulum and remains imbedded in it. The masses of the bullet and of the pendulum bob are known, shew how by observing the amplitude of the first swing of the pendulum the velocity of the bullet may be found.

11. A particle is describing simple harmonic oscillations in a straight line, shew how to determine graphically its velocity, and prove that the potential energy of the particle is proportional to the square of its distance from its equilibrium position.

12. Describe the motion of the bob of a pendulum if the upper part of the string is doubled and the two ends are attached to two separate points in the same horizontal line.

Under what circumstances does the bob describe a path analogous to that of a falling stone?

13. What are isochronous vibrations?

Shew that a body will execute isochronous vibrations if its potential energy varies as the square of its distance from its position of equilibrium.

14. The mass of a simple pendulum of length 3 ft. is 2 lb. The pendulum is raised till it makes an angle of 30° with the vertical and let go; find its energy.

15. Two equal masses are attached to the ends of a string passing through a hole in a smooth horizontal table. One mass moves uniformly on the table in a circle round the hole at its centre, while the other mass is thus kept hanging at a constant distance below the hole. Find the velocity of the moving mass if the radius of the circle be 6 yards.

16. A mass of 1 gramme moves with a harmonic motion and vibrates 128 times a second through a range of 1 cm. Find the energy it possesses.

17. A mass of m pounds is suspended from a point by a string of length l feet. The mass revolves in a horizontal circle with the string inclined to the vertical and makes n revolutions per second. Shew that the string is inclined to the vertical at an angle whose cosine is

$$\frac{g}{4\pi^2 n^2 l}.$$

MISCELLANEOUS EXAMPLES.

1. What is meant by 1 kilogramme and by 1 oz.? Describe carefully the observations you would make to determine the number of ounces in 1 kilogramme if given a balance, not known to be in adjustment, and a set of weights.

2. What is the "Standard yard"?

Describe carefully the method you would employ to determine the length of a given yard measure in terms of the Standard, and to check the accuracy of its inch divisions.

3. What is the unit of force in absolute measure? If the unit of mass be changed from a gramme to a kilogramme and the unit of time from a second to an hour, in what proportion is the measure of the weight of a given body affected?

4. What do you understand by an absolute system of units and a derived unit?

What would be the unit of time of the British absolute system if the weight of one pound at London were to be the absolute unit of force, the units of length and mass remaining unaltered?

What would be the advantages and disadvantages of such a change?

5. What is the unit of force on the C.G.S. system? and what is the unit of work? What are the respective units commonly employed in England?

6. If a nation uses 39 inches as unit length, 3 seconds as unit time, and 1 cwt. as unit mass, what is the unit force in lb. weight?

7. The measure of a certain power is 10 when 1 ft., 1 sec., and 1 lb. are the units. What is its measure when 1 yd., 1 minute, and 1 ton are the units?

8. Define velocity. What do we mean when we say that a train is travelling at the rate of 60 miles an hour?

Find the direction of the blow which drives a cricket-ball to square-leg with the same velocity as that with which it reaches the bat.

9. A stone is dropped into a well from the surface of the ground. The sound of its reaching the water is heard $5\frac{3}{8}$ seconds after the stone was dropped. If the velocity of sound is reckoned as 1000 feet per second, find the depth of the well. ($g=32$ feet per second in each second.)

10. A traveller alights from a tramcar, which is traversing a straight street, and starts to walk at 4 miles an hour along a straight side-street; after walking 10 minutes he reaches a street crossing his own at right angles, and, looking along it, he sees his tramcar at the end of it, half-a-mile away. Find the velocity of the tramcar; and draw a diagram shewing the inclination of the streets.

11. Two trains are moving on two lines inclined at a small angle to one another, with the same velocity. Shew that if an observer in one train fix his eye upon a particular point of the other train, this other train may seem to be moving faster or slower than his own, or at the same rate, according to the direction, relative to him, of the point of the other train which he is watching.

12. The horizontal velocity of a shot is 1100 feet per second and the range 3000 yards: find the initial velocity.

13. Two ships are sailing in directions making an angle of 60° with each other, with velocities of 15 and 20 miles an hour respectively. Find the magnitude of the velocity of one ship relative to the other.

14. How is the measure of an acceleration altered when the unit of time is changed from a second to a minute?

15. A train uniformly retarded has its velocity reduced from 30 to 24 miles per hour in 15 minutes. Find how far it goes in the interval.

16. Find the acceleration of a train, supposing it uniform, which passes one station at the rate of 20 miles an hour, and another 5 miles distant at the rate of 30 miles an hour.

17. A stone is thrown from a railway train with such a velocity in a direction at right angles to the path of the train that, relatively to the ground, it has a velocity of 30 miles an hour in a direction making 30° with the path of the train. What is the velocity of the train?

18. A particle moving with uniform acceleration has at a given instant a velocity of 33 feet per second; eleven seconds later it has a velocity of a mile per minute. Determine the measure of the acceleration, a foot and a second being the units of length and time.

19. With a foot and a second as units of length and time, the measure of an acceleration is 49; find the unit of time when the measure of the same acceleration is 12, and the unit of length a yard.

20. The line AB is vertical, and ACB is a right angle. Shew that the time of sliding down either AC or CB, supposed smooth, is equal to the time of falling down AB.

21. A train is moving at a rate of 60 miles an hour, and a gun is to be fired from a carriage window to hit an object which at that moment is exactly opposite the window. If the velocity of the bullet be 440 feet per second, find the direction in which the gun must be pointed.

22. A bag of ballast is dropped from a balloon when the balloon is ascending at the rate of 10 feet per second and is at a height of 300 feet; find the time occupied in the bag's descent.

(The acceleration due to gravity measured in feet and secs. may be taken as 32.)

23. A boy throws a stone 150 feet vertically upwards. What was the velocity of the stone when it left the boy's hand?

24. Compare the velocities of two trains, one of which is moving at the rate of 66 feet per second, and the other at the rate of 40 miles an hour.

25. How is uniform acceleration measured? If the measure of the acceleration due to gravity be 32, the foot and the second being the units of length and time; find its measure when 2 feet and $\frac{1}{2}$ second are the units of length and time.

26. A boat is set with her head due N.E. Under the action of the wind alone the boat would move in a N.E. direction with a velocity of $4\sqrt{2}$ miles per hour. The tide is flowing due south at a rate of 4 miles per hour. Shew that the boat's actual course is due east.

27. A cricket-ball is thrown vertically upwards with a velocity of 56 feet per second. Find the velocity when it is half-way up, and the height to which it has risen when half the time to the highest point has elapsed. (The resistance of the air is neglected, and the acceleration of gravity = 32 feet per second in each second.)

28. A man starts at right angles to the bank of a river, at the uniform rate of $1\frac{1}{2}$ miles per hour, to swim across; the current for part of the way is flowing uniformly at the rate of 1 mile per hour, and for the remainder of the way at double that rate. He finds when he reaches the other side that he has drifted down the stream a distance equal to the breadth of the river. At what point did the speed of the current change?

29. Shew that the difference of the square of the velocities at any two points, of a body falling in vacuo, varies as the distance between them.

A body falls from rest in vacuo through a certain height and acquires a certain velocity. Find how much further the body will have fallen when it has doubled its velocity.

30. A stone is thrown vertically upwards with a velocity of 36 feet per second. To what height will it rise, and after what intervals of time will it have a velocity of 12 feet per second?

31. A stone is dropped from a height of 8 feet above the ground from the window of a railway carriage travelling at the rate of 15 miles an hour; find its velocity on striking the ground.

32. If a small smooth ball be set rolling up an inclined plane in a direction other than the line of slope, find the curve which it will describe.

33. A projectile weighing half a ton is fired with a velocity of half a mile a second from a 100-ton gun. Find the velocity of recoil of the gun, and compare its kinetic energy with that of the projectile.

34. Two masses of 3 lb. and 4 lb. respectively are connected by a string passing over a pulley. Find the acceleration and tension of the string.

35. A 50-ton engine moving at the rate of 10 miles an hour impinges on a truck at rest weighing 10 tons, and the two move on together. Find their velocity and calculate the loss of kinetic energy.

36. Two equal weights of 1 lb. are connected by a fine string passing over a light pulley. A weight of 1 oz. is attached to one of them. Find the acceleration and the tension of the string.

37. A ball 1 lb. in mass, with coefficient of restitution $\frac{3}{4}$, is let fall to the ground from a height of 32 feet. Find the loss of kinetic energy on impact, and the height to which the ball will rebound.

38. In the system of pulleys in which each pulley hangs by a separate string and all the pulleys are of the same weight, find the acceleration of the weight when there are n pulleys, all the strings being vertical.

39. A fly-wheel is brought to rest after n revolutions by a constant frictional force applied tangentially to its circumference. If k be the kinetic energy of the wheel before the friction is applied and r its radius, shew that the friction is $k/2\pi nr$.

40. A lump of clay weighing 10 lb. is thrown with a velocity of 50 feet per second against an equal lump at rest: if the two travel together with a velocity of 25 feet per second, find the loss in energy estimated in foot-pounds.

41. Find the amount of work done in drawing a weight of 3 tons 100 yards along a rough horizontal plane, when the friction is 25 lb.-wt. per ton.

42. A colliery engine draws 10 tons of coal up a shaft 1000 feet deep in 1 minute. Find the total amount of work done and average power given out by the engine.

43. The handle of a hoisting-crab moves through 1 foot while the weight lifted moves through $\frac{1}{4}$ inch. It is found that to raise 1 ton a force of 80 lb.-wt. must be applied to the handle. What proportion of the work is spent in overcoming friction?

44. How much work is done in elevating 2 cwt. of coals from the bottom to the top of a staircase, the staircase having 66 steps and each step 8 inches high?

45. Two stones are thrown at the same instant from the tops of two towers directly at one another; shew that, neglecting the effect of the atmosphere, they will meet if the velocities of projection are great enough.

46. A particle is moving in a circle with constant speed. Assuming the mass and speed of the particle and the radius of the circle to be known, state the force acting on the particle and its acceleration.

47. A shot 8 lb. in mass leaves a gun 18 tons in mass with a velocity of 1500 feet per second; with what velocity does the gun recoil?

48. A constant horizontal force will give to a mass of 15 lb., starting from rest and supported on a smooth horizontal plane, a velocity of 28 feet per second in 4 seconds. What weight will that force support?

49. A force equal to the weight of 10 lb. acts for a minute on a mass of 1 cwt. Find the momentum and energy of the mass. What is the work done by the force?

50. A truck which weighs 10 tons is free to move without friction in a horizontal direction under the action of a horizontal force equal to the weight of 42 lb.; find the acceleration and determine how fast (in miles per hour) the truck would be moving if the force continued to act for an hour.

51. How fast must the bob of a pendulum (length 3 ft.) be moving at the bottom of its swing if it is to reach the horizontal through the point of support before it turns?

Will the required velocity be greater for a small bob than for a large one? Give reasons for your answer.

52. Determine the kinetic energy lost in the direct impact of two elastic spheres, each of a pound mass, moving in opposite directions, each with a velocity of 1 mile in 3 minutes; the coefficient of restitution being $\frac{1}{2}$.

53. An engine of 45 horse-power is drawing a train; if the resistance to the motion when moving with a velocity of v feet per second be $\frac{1}{18}v^2$ lb. weight, find the maximum velocity the train can attain. [1 horse-power = 550 ft.-lb. per second.]

54. Find the number of foot-pounds of work which must be done on a fly-wheel whose mass is 1000 lb. and radius 30 inches to give it a velocity of 600 revolutions per minute, assuming the whole mass of the wheel to be concentrated in the rim.

55. Shew that if a body falls freely under gravity there is neither loss nor gain of energy; and explain how to apply the same principle to find the velocity of a body sliding down a smooth curve.

EXAMINATION QUESTIONS.

I.

1. What are the units in terms of which length, mass and time are usually measured (1) in England, (2) on the Metric System?

2. Define the terms Motion, Velocity, Speed, Acceleration, and explain how they are measured.

3. What is meant by the composition of Velocities? Enunciate and prove the parallelogram of Velocities.

4. Shew that in the case of a particle moving with uniform acceleration a in a straight line the following relations hold

$$v = at \qquad s = \tfrac{1}{2}at^2 \qquad v^2 = 2as,$$

v being the velocity at the end of the time t, and s the space passed over.

5. State Newton's Laws of Motion, explaining the terms used in your statement.

6. Define Force and shew how the second law enables us to obtain a measure of force; prove the formula $F = ma$, where a is the acceleration produced in mass m by the force F. In what units must these various quantities be measured if the above formula is to hold?

II.

1. State the second law of motion and shew clearly how to use it to determine the measure of the unit of force. Define the terms dyne, poundal.

2. What experiments are required to shew that the weight of a body is proportional to its mass?

3. Describe Atwood's machine and shew how to use it to verify the formulae $s = vt$, $v = at$, $s = \frac{1}{2}at^2$, $F = ma$, with the usual notation.

4. How would you verify the fact that the time of fall of a body is independent of its horizontal motion and that the path of a projectile is a parabola?

5. What is meant by a simple pendulum? Shew that the time of swing of such a pendulum is proportional to the square root of its length. What inference do you draw from the fact that it does not depend on the mass of the bob?

6. State the third law of motion.

Define the terms work and energy; and shew that if a body of mass m has a velocity v produced in it by the action of a constant force F, then the work done by the force is $\frac{1}{2}mv^2$.

7. Shew that if a body is falling freely under gravity its energy remains constant.

III.

1. Explain the terms Work and Energy and shew how work is measured; distinguish between an erg, a foot-pound and a foot-poundal.

What do you understand by Power? What is a Horse-power?

2. Distinguish between kinetic and potential energy and shew how they are measured in the case of a falling body.

Shew that if a body be let fall from rest its energy remains constant till it reaches the ground.

3. State and prove the parallelogram of forces.

4. What is meant by the Resolution of forces? Shew how to find the resolved part of a force in two directions at right angles.

5. A body is projected in a horizontal direction; shew that its path is a parabola and find its latus rectum.

6. Describe experiments to determine the relation between the relative velocities before and after impact of two balls which impinge directly.

7. Shew that a body moving with uniform speed v in a circle of radius r has acceleration equal to v^2/r.

8. Explain what is meant by simple harmonic motion.

ANSWERS TO EXAMPLES.

CHAPTER I. (Page 18.)

1. (1) 35·558 cms.; (2) 182·87 cms.; (3) 152·39 cms.; (4) 20115·82 cms.

2. (1) 1·60927; (2) 6437·08; (3) ·5486; (4) ·001; (5) ·000025.

3. (1) 191·58; (2) 5747·38; (3) 4046,153,000; (4) 471,428·6.

4. (1) 2919·52; (2) 2627570; (3) 1302266×10^{12}; (4) 1768×10^{7}.

5. 56π sq. in. = 176 sq. in. **6.** $\frac{1}{4\pi}$ sq. miles = 246,400 sq. yds.

7. 42 ft. **8.** The circle. **9.** $25\sqrt{3}$ sq. ft. = 43·3 sq. ft.

10. $\frac{20\sqrt{2}}{\sqrt{3}}$ cm. and $\frac{20}{\sqrt{3}}$ cm. **11.** 800 sq. cm.

12. 2·10 mm., 55·44 sq. mm. **13.** ·428 mm.

14. 112·62 grms. **15.** 76·37 lb. per c. foot.

16. 1·19 grms. per c. cm. **17.** 1·22 grms. per c. cm.

18. 18·5 grms. per c. cm. **19.** 3437 grains per c. inch.

20. The density of the sphere = the density of the cylinder $\times$ 7·80.

CHAPTER II. (Page 53.)

1. (1) $14\frac{2}{3}$; (2) 30; (3) $981,333,333\frac{1}{3}$; (4) $1527\frac{7}{9}$.

2. (1) $\frac{a}{b}$; (2) $62\frac{6}{7}$; (3) $\frac{19}{150}$ = ·1267; (4) 2·032.

3. 1117·545. **4.** (1) 2640; (2) 3801600; (3) 1387584000.

5. (1) $8\frac{8}{21}$ ft. per sec.; (2) $5\frac{5}{7}$ miles per hour.

6. 4 miles from A's starting-place.
A's speed is 4·8 miles per hour.
B's „ 2·4 „ „

7. 5 miles. **8.** 2750 ft. **9.** $\frac{5}{8}$ ft. per second.

10. 248 to 256 approximately. **11.** (1) 5; (2) 10; (3) 19·21; (4) 5.

12. (1) $\sqrt{37}=6{\cdot}083$; (2) 12·96; (3) 2·91; (4) 2·65.

13. 5 miles per hour. **14.** 16·1 ft. per second.

15. 45 minutes from start. 8 miles.

18. (1) $\sqrt{2}$; (2) $2\sqrt{2}$; (3) $2\sqrt{5}=4{\cdot}47$; (4) $3\sqrt{2}=4{\cdot}24$.

19. (1) 0; (2) $\sqrt{3}$; (3) $\sqrt{19}=4{\cdot}36$; (4) $3\sqrt{3}$.

20. (1) $500\sqrt{3}$ horizontally, 500 vertically;
$500\sqrt{2}$,, $500\sqrt{2}$,,
500 ,, $500\sqrt{3}$,,

(2) $\dfrac{55}{\sqrt{3}}=31{\cdot}8$ ft. per sec. horizontally, $18\frac{1}{3}$ ft. per sec. vertically.

21. 577 ft. per sec. **22.** $250\sqrt{3}$ ft. per sec. $=433$ ft. per sec.

23. $5\sqrt{2}$ miles per hour in the north-west direction. **24.** *BC*.

25. (*a*) $35\sqrt{2}$ cms. per sec. to the north-west; (*b*) 6.

26. $30\sqrt{3}$ miles per hour. **27.** 900 ft.

28. The velocities during each of the 4 minutes are 2, 6, 10 and 14 ft. per second respectively.

29. $\dfrac{t^2}{2}$ ft. **30.** The angle between the two components is $\cos^{-1}=\frac{1}{15}$.

32. $\frac{1}{2}AB$. **33.** $nu\tau+v\tau^2\dfrac{n(n-1)}{2}$. **34.** 73·3 to 55.

CHAPTER III. (Page 73.)

1. $\frac{16}{3}g$ and $\frac{12}{7}g$. **2.** $\frac{3}{2}g$. **3.** 15625 ft. 62·5 sec.

4. (1) 12·5; (ii) 26·25. **5.** 15·625.

6. Change in velocity $=5\sqrt{2}$. Acceleration $=\dfrac{1}{\sqrt{2}}$.

7. 1150 ft. **8.** $a=\frac{44}{135}$ ($=\cdot326$) ft. per sec. per sec.

9. $u=160\sqrt{13}$ ($=577$) ft. per sec. Height $=5200$ ft.

10. 420000 ft. 1300 ft. per sec. **11.** 36 sec.

12. (*a*) 3 sec. $13\frac{1}{2}g$. (*b*) $1\frac{1}{2}$ sec. $3\frac{3}{8}g$.

13. $25\left(1-\dfrac{1}{\sqrt{2}}\right)=7{\cdot}3$ sec. $\dfrac{25}{\sqrt{2}}=17{\cdot}7$. **14.** $\dfrac{u}{a}$.

15. Half the height. $\sqrt{\text{height}\times g}$. **16.** 3 sec.

17. 272 ft. per sec. **18.** Initial velocity $=0$. **19.** 976·56 ft.
20. 4 sec. **21.** $833\frac{1}{3}$ cm. per sec. $3\frac{17}{36}$ cm. per sec. per sec.
22. 4587·2 metres. After 61·16 secs. **23.** 88·6 metres per sec.
24. 38624. **25.** 10·2 sec. **26.** 144 ft.
27. $\frac{11}{16}$ ft. per sec. per sec. **28.** $12\frac{2}{3}$ ft. per sec. No.
29. 16 ft. per sec. $a=\frac{2}{15}$ ft. per sec. per sec.
30. $-\frac{121}{2160}$ $(=-{\cdot}056)$ ft. per sec. per sec. **31.** $13\frac{3}{4}$ secs.
34. 25 ft. 40 ft. per sec. **36.** $61\frac{5}{11}$ ft. per sec. $276\frac{6}{11}$ ft.
37. velocity at end of 5 sec. $=320$ ft. per sec.
initial velocity $=20$ ft. per sec.
38. 120 ft. **39.** 120 ft.
40. At an angle $\cos^{-1}\frac{1}{5}$ with the direction of motion of the train.
41. $6\frac{3}{4}$ miles per hour. $35\frac{1}{5}$ ft. per sec.
1 mile in $6\frac{1}{5}$ min. $=14{\cdot}2$ ft. per sec. **42.** tangent $=\frac{5}{4}$.
43. velocity $=\dfrac{3\pi}{10}$ ft. per sec.; angular velocity $=\dfrac{\pi}{10}=(18^\circ)$ per sec.
44. mean acceleration $=\dfrac{v-u}{t}$. **46.** 100 ft. 80 ft. per sec.
47. $32\sqrt{5}$ ft. per sec. $\sqrt{5}$ secs.
48. 14 ft. per sec. 24 ft. per sec. per sec. **49.** 159 ft. approx.
50. 3888000 miles per hr. per hr.

CHAPTER VII. (Page 142.)

1. (i) 100,000 units of momentum;
(ii) $20\frac{5}{8}$; (iii) 2,682,240; (iv) 9.900.000.
2. momentum of second mass $=$ momentum of first mass $\times\, 231{\cdot}5$.
3. $27\frac{7}{9}$ cm. per sec. per sec. $1,666,666\frac{2}{3}$ dynes.
4. impressed force on second mass
$=$ impressed force on first mass $\times\, {\cdot}2572$.
5. $469{\cdot}5\pi \times 10^{31}$ c.g.s. units of momentum. **6.** 45·4 cm.
7. impulse $=30\sqrt{5}\times m$, where $m=$ mass of cricket-ball;
average force $=1500\sqrt{5}\times m$ poundals.
8. 2215 cm. per sec. **9.** 256 lbs. 8192 poundals. **11.** $62\frac{1}{2}$ poundals.
12. $363\frac{17}{27}$ cm. per sec. $181\frac{23}{27}$ cm. $21822\frac{2}{3}$ cm. **13.** 24 ft. per sec.
14. $10\frac{2}{3}$ pounds. **15.** $21\frac{9}{11}$ pounds. **16.** 60,500 ft. $=11\frac{11}{24}$ miles.
17. $-1\frac{47}{75}$ ft. per sec. per sec. $\frac{11}{2400}$ of the weight.
20. forces are equal. **21.** 200 poundals. **22.** $2133\frac{1}{3}$ poundals.
23. a dyne. **25.** $17\frac{7}{9}$ poundals. $28\frac{4}{5}$ ft. **26.** $5\sqrt{30}$ mm. $=27{\cdot}4$ mm.

27. $\frac{80\sqrt{5}}{3} = 59 \cdot 6$ ft. per sec. **28.** 16 ft. per sec.

29. 255·7 cm. per sec. **30.** 38080 poundals.

31. $2\frac{2}{3}$ lb. weight. $10\frac{2}{3}$ ft. per sec. per sec.

32. 38·8 ft. per sec. per sec. **33.** 14·85 cm. per sec. per sec.

34. $5\frac{1}{3}$ ft. per sec. per sec. 16 ft. per sec. 24 ft.

35. 8·83 cm. per sec. per sec.

36. (1) $\frac{32h}{l}$ ft. per sec. per sec. (2) $\frac{l}{4\sqrt{h}}$ secs. (3) 16 ft. per sec.

37. (1) $\frac{g \cdot h}{l}$. (2) $\frac{l\sqrt{2}}{\sqrt{h \cdot g}}$ secs. (3) 28 ft. per sec.

41. $18\frac{1}{3}$. **42.** 2. **43.** $\left(\frac{n'-n}{n'+1}\right)g$.

44. (1) $2mu \sin \frac{1}{2}a$. (2) $\frac{P}{2m} - \mu g$. (3) $\frac{P}{2} + \mu mg$.

45. 313 cm. per sec. **46.** (1) 30°. (2) 56·6 cm. per sec.

47. (1) $2\frac{2}{3}$ ft. per sec. per sec. (2) 24 F.P.S. units of momentum.

48. (1) $5\frac{1}{3}$ ft. per sec. per sec. (2) $106\frac{2}{3}$ poundals. (3) $66\frac{2}{3}$ ft.

49. (1) 3 tons' wt. (2) 525·2 ft. **50.** 1 kilo. wt.

51. 20 stone wt. **52.** $288\sqrt{2}$ ft.

54. 45 poundals. **55.** 125 ft.

56. (1) 2 ft. per sec. per sec. (2) 150 poundals and 136 poundals.

57. (1) $\frac{2 \cdot m' \cdot m \cdot g}{m' + m}$. (2) $1\frac{1}{2}$ secs.

CHAPTER VIII. (Page 184.)

3. $\frac{10^4}{g^2} = \cdot 0104$ secs.

4. (1) momentum of bullet = 37·5 units;
energy ,, = 22,500 foot-poundals.

(2) momentum of large mass = 560 units;
energy ,, ,, = 140 foot-poundals;
force to stop bullet in $\frac{1}{10}$ sec. = 375 poundals;
,, ,, large mass ,, ,, = 5600 ,,
work done by bullet = 22,500 foot-poundals;
,, ,, large mass = 140 ,,

6. $150g \,.\, \sqrt{2}$ ergs. 8. 9 cm. per sec.; before impact = 3840 ergs.
after „ = 3240 „

9. (1) 1792 ft. per sec. (2) $16\sqrt{7} = 42 \cdot 4$ ft. per sec.

10. (1) 19,200 F.P.S. units of momentum.
(2) 1645714$\frac{2}{7}$ ft.-poundals. (3) 1645714$\frac{2}{7}$ ft.-poundals.

11. (1) 31,250 ft.-poundals. (2) 10,416$\frac{2}{3}$ poundals. 12. 149 oz. 2·5 ft.-tons.

13. $\frac{4}{15}$ horse-power. 14. 5 ft.-pounds. 15. 5·7 horse-power.

16. When the pendulum has fallen through half the vertical height of its swing.

17. 5,040,000 ft.-poundals. 18. 3520 ft.-pounds.

19. (1) 20 ft. per sec. (2) 1 to 100.

20. 17$\frac{1}{7}$ ft. per sec. 21. 1·59 H.P. 22. 74$\frac{2}{3}$ H.P.

23. 2199$\frac{3}{11}$ H.P. 25. 4 inches. 28. 7812$\frac{1}{2}$ poundals.

29. 31,500 poundals. 30. 2685×10^{10} ergs. 31. 68$\frac{3}{4}$ c. ft.

CHAPTER IX. (Page 208.)

1. $40\sqrt{2}$ yds. 2. 14·06 ft. 4. (1) 108 ft. (2) 3 secs.

5. (1) 1000 ft. per sec. (2) 144 ft. 6. (1) 13 secs. (2) $104g$ ft.

7. $\dfrac{55 \sin 2\alpha}{12}$ miles. 9. 100 ft. per sec. 10. 70 ft. per sec.

12. (1) 22,050 ft. (2) 553·48 ft. per sec. 14. $\dfrac{-V^2 \cos 2\alpha}{2g}$.

CHAPTER X. (Page 221.)

1. $\frac{2}{5}$ of that of the impinging ball. 2. 2 to 5. $\frac{3}{4}$.

4. 15 ft. per sec. in opposite directions.

5. Their masses are equal and e is unity. 6. ·0878 ft.

7. $\tan^{-1}\frac{4}{3}$. 8. $v = \frac{1}{2}u$; $\beta = 90°$. $v' = \dfrac{\sqrt{3}}{2}u$; $\beta' = 0°$.

9. $\cos \alpha = \dfrac{v}{2u}$. 11. $\dfrac{1-e^2}{4} \cdot u$.

CHAPTER XI. (Page 236.)

2. 1·65 revolutions per second. **3.** lengthened by ·08 inches.

5. (1) In first position pressure on support $= 800\sqrt{\frac{5}{8}}$ grms. wt.,
tension of string $= 200\sqrt{\frac{5}{8}}$ „
weight of bob (1000 grms. wt.).

(2) In second position pressure on support $250\sqrt{15}$ grms. wt.

(3) In third position pressure on support $\left(1+\dfrac{1}{2\sqrt{15}}\right)$ kilos. wt.

7. Intensity of gravity at B = intensity at $A \times 1{\cdot}00023$.

14. ·804 ft.-pounds. **15.** 24 ft. per sec. **16.** 80854 ergs.

23. $40\sqrt{6}$ ft. per sec. **24.** 9 to 8. **25.** 1.

27. $28\sqrt{2}$ ft. per sec. $36\frac{3}{4}$ ft. **28.** $\frac{3}{4}$ of the way across.

29. 3 times as far as it fell in the first case.

30. $20\frac{1}{4}$ ft. After $\frac{3}{4}$ sec. and $1\frac{1}{2}$ secs. **31.** 31·6 ft. per sec.

33. (1) $13\frac{1}{3}$ ft. per sec.

(2) kinetic energy of gun $= \dfrac{\text{kinetic energy of projectile}}{200}$.

34. (1) $4\frac{4}{7}$ ft. per sec. per sec. (2) $109\frac{5}{7}$ poundals.

35. (1) $8\frac{1}{3}$ miles per hour. (2) $933{,}333\frac{1}{3}$ ft.-poundals.

36. (1) $\frac{32}{33}$ ft. per sec. per sec. (2) 33 poundals.

MISCELLANEOUS EXAMPLES. (Page 238.)

3. Increased 12,960 times. **4.** $\dfrac{1}{\sqrt{g}}$ secs.

6. $1\frac{19}{72}$ lb.-wt. **7.** $\frac{14}{15}$.

8. 45° to the direction in which the ball is coming. **9.** 400 ft.

10. 7 miles per hour. **12.** 1108 ft. per sec.

13. $5\sqrt{13}$ miles per hour. **14.** Increased 3600 times.

15. $6\frac{3}{4}$ miles. **16.** $\frac{11}{540}$ ft. per sec. per sec.

17. $15\sqrt{3}$ miles per hour. **18.** 5 ft. per sec. per sec.

19. $\frac{6}{7}$ secs. **21.** $\frac{\pi}{2}+\sin^{-1}\frac{1}{3}$, with the motion of the train.

22. 4·65 secs. **37.** (1) 448 ft.-poundals. (2) 18 ft.

40. $195\frac{5}{16}$ ft.-lb. **41.** 22500 ft.-lb.

42. (1) 22400000 ft.-lb. (2) 678·8 H.P. **43.** $\frac{5}{12}$.

44. 9856 ft.-lb. **47.** $\frac{25}{34}$ ft. per sec. **48.** $3\frac{9}{32}$ pounds.

49. (1) 19200 F.P.S. units of momentum. (2) $1648571\frac{3}{7}$ ft.-poundals.
(3) $1648571\frac{3}{7}$ ft.-poundals.

50. (1) $\frac{3}{50}$ ft. per sec. per sec. (2) $147\frac{3}{11}$ miles per hour.

51. $8\sqrt{6}$ ft. per sec. **52.** 11 ft.-poundals. **54.** 38554687·5 ft.-lb.

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